T-DUALITY AND HOMOLOGICAL MIRROR SYMMETRY OF TORIC VARIETIES

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Abstract. Let X_Σ be a complete toric variety. The coherent-constructible correspondence κ of [FLTZ] equates $\mathcal{P}\mathrm{erf}_T(X_\Sigma)$ with a subcategory $Sh_{cc}(M_\mathbb{R};\Lambda_\Sigma)$ of constructible sheaves on a vector space $M_\mathbb{R}$. The microlocalization equivalence μ of [NZ, N1] relates these sheaves to a subcategory $Fuk(T^*M_\mathbb{R};\Lambda_\Sigma)$ of the Fukaya category of the cotangent $T^*M_\mathbb{R}$. When X_Σ is nonsingular, taking the derived category yields an equivariant version of homological mirror symmetry, $DCoh_T(X_\Sigma)\cong DFuk(T^*M_\mathbb{R};\Lambda_\Sigma)$, which is an equivalence of triangulated tensor categories.

The nonequivariant coherent-constructible correspondence $\bar\kappa$ of [Tr] embeds $\mathcal{P}\mathrm{erf}(X_\Sigma)$ into a subcategory $Sh_c(T^\vee_\mathbb{R};\bar\Lambda_\Sigma)$ of constructible sheaves on a compact torus $T^\vee_\mathbb{R}$. When X_Σ is nonsingular, the composition of $\bar\kappa$ and microlocalization yields a version of homological mirror symmetry, $DCoh(X_\Sigma)\hookrightarrow DFuk(T^*T_\mathbb{R};\bar\Lambda_\Sigma)$, which is a full embedding of triangulated tensor categories.

When X_Σ is nonsingular and projective, the composition $\tau=\mu\circ\kappa$ is compatible with T-duality, in the following sense. An equivariant ample line bundle $\mathcal{L}$ has a hermitian metric invariant under the real torus, whose connection defines a family of flat line bundles over the real torus orbits. This data produces a T-dual Lagrangian brane $\mathbb{L}$ on the universal cover $T^*M_\mathbb{R}$ of the dual real torus fibration. We prove $\mathbb{L}\cong\tau(\mathcal{L})$ in $Fuk(T^*M_\mathbb{R};\Lambda_\Sigma)$. Thus, equivariant homological mirror symmetry is determined by T-duality.

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1. Introduction

In this paper we derive equivariant and nonequivariant versions of the homological mirror symmetry for nonsingular complete toric varieties from the coherent-constructible correspondence [FLTZ, Tr] and microlocalization [NZ, N1]. The composition of the coherent-constructible correspondence and microlocalization sends an equivariant (resp. nonequivariant) coherent sheaf on the toric variety to an object in the Fukaya category of the cotangent bundle of a vector space (resp. a compact torus).

For nonsingular projective toric varieties, the equivariant homological mirror symmetry is determined by equivariant ample line bundles. We prove that the image of an equivariant ample line bundle agrees up to isomorphism with the Lagrangian constructed by T-duality.

1.1. Main Results.

In this paper, we work over $\mathbb{C}$. Let X_Σ be an n-dimensional complete toric variety defined by a finite complete fan $\Sigma \subset N_\mathbb{R}$. Then $T \cong (\mathbb{C}^*)^n$ acts on X_Σ, and $N_\mathbb{R} \cong \mathbb{R}^n$ can be identified with the Lie algebra of the maximal compact subgroup $T_\mathbb{R} \cong U(1)^n$. Let $M_\mathbb{R} = \mathrm{Hom}_\mathbb{R}(N_\mathbb{R}, \mathbb{R}) \cong \mathbb{R}^n$ be the dual real vector space of $N_\mathbb{R}$. Then the lattice $M = \mathrm{Hom}(T_\mathbb{R}, U(1)) \cong \mathbb{Z}^n$ naturally sits in $M_\mathbb{R}$, and the quotient $T^\vee_\mathbb{R} = M_\mathbb{R}/M$ is the dual torus of $T_\mathbb{R}$.

The first main theorem in this paper is the following:

Theorem 1. *Let X_Σ be a complete toric variety defined by a finite complete fan $\Sigma \subset N_\mathbb{R}$. Then there is a quasi-equivalence of A_∞ categories:*

$$\tau : \mathcal{P}\mathrm{erf}_T(X_\Sigma) \xrightarrow{\cong} Fuk(T^*M_\mathbb{R}; \Lambda_\Sigma). \tag{1}$$

This functor intertwines the usual monoidal product on $\mathcal{P}\mathrm{erf}_T(X_\Sigma)$ and a product structure $\diamond$ on $Fuk(T^*M_\mathbb{R}; \Lambda_\Sigma)$ up to a quasi-isomorphism.

In (1), $\mathcal{P}\mathrm{erf}_T(X_\Sigma)$ is the dg category of equivariant perfect complexes on X_Σ (see Section 2.2 for the precise definition), and $Fuk(T^*M_\mathbb{R}; \Lambda_\Sigma)$ is a subcategory of the unwrapped Fukaya category $Fuk(T^*M_\mathbb{R})$ determined by the fan Σ (see Section 3 for the precise definition). As we will explain in Section 3, Theorem 1 follows from the results in [FLTZ, NZ, N1]. We use the results in [N1] to define the product $\diamond$ on $Fuk(T^*M_\mathbb{R})$ (actually any cotangent bundle of a Lie group); for cotangent

fibers we have $T_{x_1}M_{\mathbb{R}} \diamond T_{x_2}M_{\mathbb{R}} = T_{x_1+x_2}M_{\mathbb{R}}$. The monoidal product structure on $\mathcal{P}\mathrm{erf}_T(X_\Sigma)$ comes from the usual tensor product of vector bundles.

When X is nonsingular, taking H^0 of (1) yields the following:

Corollary 1 (equivariant homological mirror symmetry of toric varieties)**.** *Let X_Σ be a nonsingular complete toric variety defined by a nonsingular finite complete fan $\Sigma \subset N_{\mathbb{R}}$. (In particular, X_Σ is a compact complex manifold.) Then there is an equivalence of tensor triangulated categories:*

$$H(\tau) : DCoh_T(X_\Sigma) \xrightarrow{\cong} DFuk(T^*M_{\mathbb{R}}; \Lambda_\Sigma). \tag{2}$$

In (2), $DCoh_T(X_\Sigma)$ is the bounded derived category of equivariant coherent sheaves on X_Σ. The equivalence (2) preserves the tensor product, so it is a stronger equivalence than the equivalence in the usual homological mirror symmetry. Note that we do not assume X_Σ is projective in Corollary 1, so a priori the other direction of homological mirror symmetry (involving the Fukaya category of the toric variety) does not make sense.

Our second main theorem concerns the nonequivariant version of Theorem 1:

Theorem 2. *Let X_Σ be a complete toric variety defined by a finite complete fan $\Sigma \subset N_{\mathbb{R}}$. Then there is a quasi-embedding of A_∞ categories:*

$$\bar{\tau} : \mathcal{P}\mathrm{erf}(X_\Sigma) \longrightarrow Fuk(T^*T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma). \tag{3}$$

The functor $\bar{\tau}$ intertwines the product $\diamond$ on $Fuk(T^*T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma)$ and the usual monoidal product on $\mathcal{P}\mathrm{erf}(X_\Sigma)$.

In (3), $\mathcal{P}\mathrm{erf}(X_\Sigma)$ is the dg category of perfect complexes on X_Σ (see Section 2.2 for the precise definition), and $Fuk(T^*T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma)$ is a subcategory of the unwrapped Fukaya category $Fuk(T^*T^\vee_{\mathbb{R}})$ determined by the fan Σ (see Section 3 for the precise definition). The diamond product $\diamond$ is similarly defined as in the equivariant case from the Lie group structure on $T^\vee_{\mathbb{R}}$. As we will explain in Section 3, Theorem 2 follows from the results in [Tr, NZ, N1]. We conjecture that (3) is a quasi-equivalence.

When X is nonsingular, taking H^0 of (3) yields the following:

Corollary 2 (homological mirror symmetry for toric varieties)**.** *Let X_Σ be a nonsingular complete toric variety defined by a nonsingular finite complete fan $\Sigma \subset N_{\mathbb{R}}$. Then there is a full embedding of tensor triangulated categories:*

$$H^0(\bar{\tau}) : DCoh(X_\Sigma) \longrightarrow DFuk(T^*T^\vee_{\mathbb{R}}; \bar{\Lambda}_\Sigma) \tag{4}$$

In (4), $DCoh(X_\Sigma)$ is the bounded derived category of coherent sheaves on X_Σ. We conjecture that (4) is an equivalence. In Appendix C, we will comment on the relationships among the Fukaya categories in Theorem 1, the physical/traditional mirror of toric varieties (Landau-Ginzburg/Fukaya-Seidel category), and the relative Fukaya category in Abouzaid's work [Ab1, Ab2], when X_Σ is a nonsingular projective toric variety.

Our third main theorem relates Theorem 1 to T-duality. When X_Σ is nonsingular and projective, we perform an equivariant version of T-duality: for any equivariant line bundle $\mathcal{L}_{\vec{c}}$ with a $T_{\mathbb{R}}$-invariant hermitian metric h, we construct a Lagrangian $\mathbb{L}_{\vec{c},h} \subset T^*M_{\mathbb{R}}$, which projects to a Lagrangian $\bar{\mathbb{L}}_{\vec{c},h} \subset T^*T^\vee_{\mathbb{R}}$. We prove the following:

Theorem 3 (equivariant homological mirror symmetry is T-duality)**.** *Let X_Σ be a nonsingular projective toric variety defined by a fan $\Sigma \subset N_{\mathbb{R}}$. For any equivariant ample line bundle $\mathcal{L}_{\vec{c}}$ with an admissible*[1] *hermitian metric h, the T-dual Lagrangian $\mathbb{L}_{\vec{c},h}$ (constructed in Section 4) is an object in $Fuk(T^*M_{\mathbb{R}};\Lambda_\Sigma)$ and*

$$\mathbb{L}_{\vec{c},h} \cong \tau(\mathcal{L}_{\vec{c}}),$$

where τ is as in Theorem 1.

By Theorem B.2, when X_Σ is nonsingular and projective, $\mathcal{P}\mathrm{erf}_T(X_\Sigma)$ is generated by equivariant ample line bundles. Therefore equivariant homological mirror symmetry (2) is determined by T-duality. Theorem 1 and Theorem 3 imply the following.

Corollary 3 (subcategory generated by T-dual Lagrangians)**.** *Let X_Σ be a nonsingular projective toric variety defined by a fan $\Sigma \subset M_{\mathbb{R}}$. Then $Fuk(T^*M_{\mathbb{R}};\Lambda_\Sigma)$ is generated by the T-dual Lagrangians $\mathbb{L}_{\vec{c},h}$ of equivariant ample line bundles $\mathcal{L}_{\vec{c}}$ on X_Σ.*

1.2. **Simple Example.** The simple example of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is instructive. The $\mathbb{C}^*$ action is $t: z \mapsto t\cdot z$. Write $z = e^{y+\sqrt{-1}\theta}$, so $\theta \in S^1$ coordinatizes the real torus orbit. The divisors $p_0 = 0$ and $p_\infty = \infty$ span the equivariant Picard group. The equivariant line bundle $\mathcal{O}_{\mathbb{P}^1}(ap_0 + bp_\infty)$, $a, b \in \mathbb{Z}$, admits an S^1-invariant hermitian metric $h = \frac{|z|^{2b}}{(1+|z|^2)^{a+b}}$ and associated connection 1-form $A = \frac{1}{\sqrt{-1}}\partial_y \log h\, d\theta$. On each real torus $y = const$, this connection has monodromy determined by the value of $\gamma = -\partial_y \log h|_y$, a coordinate on the dual S^1. Letting y vary determines a submanifold $\mathbb{L} = \{(y,\gamma) \mid \gamma = -\partial_y \log h\} \subset \mathbb{R}^2$. By the explicit form of h, we find $\gamma = \frac{(a+b)e^{2y}}{1+e^{2y}} - b$. The *non*equivariant bundle is $\mathcal{O}_{\mathbb{P}^1}(a+b)$, and note that keeping the sum $a+b$ fixed and varying b amounts to lattice translations in the universal cover $\mathbb{R}$ of the dual torus S^1. Inverting equations, we can write $\mathbb{L}$ as a graph over an interval over length $|a+b|$ in $\mathbb{R}$, which corresponds to a constructible sheaf by [NZ].

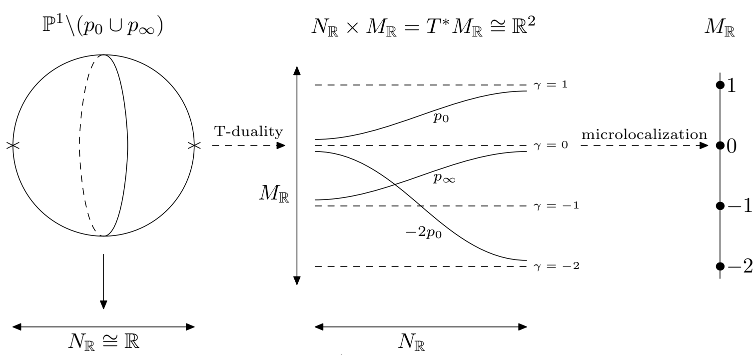

Fig.1 Our procedure for $\mathbb{P}^1$. The three Lagrangians shown above come from equivariant line bundles $\mathcal{O}_{\mathbb{P}^1}(p_0)$, $\mathcal{O}_{\mathbb{P}^1}(p_\infty)$ and $\mathcal{O}_{\mathbb{P}^1}(-2p_0)$. By microlocalization [NZ], they correspond (up to shifts) to three constructible sheaves on $\mathbb{R}$: $i_{(0,1)!}\mathbb{C}_{(0,1)}$, $i_{(-1,0)!}\mathbb{C}_{(-1,0)}$, and $i_{(-2,0)*}\mathbb{C}_{(-2,0)}$, respectively, where i is the inclusion of the indicated open interval into $M_{\mathbb{R}} \cong \mathbb{R}$.

[1] A hermitian metric h on an ample line bundle is *admissible* if it is real analytic, $T_{\mathbb{R}}$-invariant, and defines a unitary connection whose curvature is a nondegenerate closed 2-form.

1.3. **Relation to the work of others.** The present work is much related to results of several authors. Below are some comparisons; further details are given in Appendix C.

Homological mirror symmetry for toric Fano varieties was conjectured by Kontsevich [K2]. A physical proof of mirror symmetry was given by Hori-Vafa [HV]. The mirror of a toric Fano manifold is a Landau-Ginzburg model $((\mathbb{C}^*)^n, W)$ where the superpotential $W : (\mathbb{C}^*)^n \to \mathbb{C}$ is a holomorphic function. The homological mirror conjecture states (in one direction) that the derived category of coherent sheaves on the toric Fano manifold is equivalent to the derived Fukaya-Seidel category $FS((\mathbb{C}^*)^n, W)$ of the Picard-Lefschetz fibration defined by W.

Seidel proves homological mirror symmetry for $\mathbb{P}^2$ in [S1]. Auroux-Katzarkov-Orlov prove it for weighted projective planes and their noncommutative deformations in [AKO1], and for (not necessarily toric) del Pezzo surfaces in [AKO2]. Ueda proves it for toric del Pezzo surfaces [U]; Ueda-Yamazaki prove it for toric orbifolds of toric del Pezzo surfaces. Bondal and Ruan [BR] announced a proof of homological mirror symmetry for weighted projective spaces, generalizing the result by Auroux-Katzarkov-Orlov on weighted projective planes [AKO1].

The version here is somewhat different, but conjecturally related (see Section C.2) and much closer to Abouzaid's work [Ab1, Ab2]. Torus equivariance is encoded in the $\mathbb{Z}^n$ grading of morphisms in various categories introduced in [Ab2].

Recently, Subotic constructed a monoidal structure on the extended Fukaya category of any Lagrangian torus fibration with a section [Su].

We also mention some complementary work which studies the A-model for toric varieties [CO, FOOO1, FOOO2] and varieties with effective anticanonical divisors [Au], and the relation to the Landau-Ginzburg mirror—especially Chan-Leung [CL], which employs similar T-duality reasoning.

1.4. **Outline.** Section 2 contains notation and conventions for categories, sheaves and toric varieties. In Section 3, we derive Theorem 1 and Theorem 2. In Section 4, we perform an equivariant version of the T-duality, and relate the resulting T-dual Lagrangians to classical objects in symplectic geometry. In Section 5, we prove Theorem 3. Appendix A contains a brief review of analytic-geometric categories and a proof of Proposition 5.6. We show that $\mathcal{P}\mathrm{erf}_T(X)$ is generated by T-equivariant ample line bundles in Appendix B. We discuss the relation to the work of others in Appendix C.

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2. Notation and Convention

2.1. **Categories.** Throughout, we consider dg and more generally A_∞ categories. Unless otherwise stated, a category C is assumed to be completed to its triangulated

envelope. (Recall that dg and A_∞ categories have canonical triangulated structures[2] and completions[3] [S2].)

2.2. **Schemes and coherent sheaves.** All schemes that appear will be over $\mathbb{C}$. If X is a scheme, then we let $\mathcal{Q}^{naive}$ denote the dg category of bounded complexes of quasicoherent sheaves on X, and we let $\mathcal{Q}(X)$ denote the localization of this category with respect to acyclic complexes (see [Dr] for localizations of dg categories). If G is an algebraic group acting on X, we let $\mathcal{Q}_G(X)^{naive}$ denote the dg category of complexes of G-equivariant quasicoherent sheaves. We let $\mathcal{Q}_G(X)$ denote the localization of this category with respect to acyclic complexes. We use $\mathcal{P}\mathrm{erf}(X) \subset \mathcal{Q}(X)$ and $\mathcal{P}\mathrm{erf}_G(X) \subset \mathcal{Q}_G(X)$ to denote the full dg subcategories consisting of *perfect* objects—that is, objects which are quasi-isomorphic to bounded complexes of vector bundles. If $u : X \to Y$ is a morphism of schemes, we have natural dg functors $u_* : \mathcal{Q}(X) \to \mathcal{Q}(Y)$ and $u^* : \mathcal{Q}(Y) \to \mathcal{Q}(X)$. Note that the functor u^* carries $\mathcal{P}\mathrm{erf}(Y)$ to $\mathcal{P}\mathrm{erf}(X)$. Suppose G and H are algebraic groups, X is a scheme with a G-action, and Y is a scheme with an H-action. If a morphism $u : X \to Y$ is equivariant with respect to a homomorphism of groups $\phi : G \to H$, then we will often abuse notation and write u_* and u^* for the equivariant pushforward and pullback functors $u_* : \mathcal{Q}_G(X) \to \mathcal{Q}_H(Y)$ and $u^* : \mathcal{Q}_H(Y) \to \mathcal{Q}_G(X)$.

2.3. **Constructible and microlocal geometry.** We refer to [KS] for the microlocal theory of sheaves. If X is a topological space, let $Sh(X)$ denote the dg category of bounded chain complexes of sheaves of $\mathbb{C}$-vector spaces on X, localized with respect to acyclic complexes. If X is a real-analytic manifold, $Sh_c(X)$ denotes the full subcategory of $Sh(X)$ of objects whose cohomology sheaves are constructible with respect to a real-analytic Whitney stratification of X. If X is a (possibly noncompact) real-analytic manifold, then $Sh_{cc}(X) \subset Sh_c(X)$ is the full subcategory of objects which have compact support. We continue to use the phrase "sheaf" for an object of $Sh_{cc}(X)$.

The *standard constructible sheaf* on a submanifold $i : Y \hookrightarrow X$ is defined as the push-forward of the constant sheaf on Y, i.e. $i_*\mathbb{C}_Y$ as an object in $Sh_c(X)$. The Verdier duality functor $\mathcal{D} : Sh_c^\circ(X) \to Sh_c(X)$ takes $i_*\mathbb{C}_Y$ to the *costandard constructible sheaf* on X. We know $\mathcal{D}(i_*\mathbb{C}_Y) = i_!\mathcal{D}(\mathbb{C}_Y) = i_!\omega_Y$, where $\omega_Y = \mathcal{D}(\mathbb{C}_Y) = \mathbb{C}_Y[\dim Y]$.

We denote the singular support of a complex of sheaves F by $SS(F) \subset T^*X$. If X is a real-analytic manifold and $\Lambda \subset T^*X$ is an $\mathbb{R}_{>0}$-invariant Lagrangian subvariety, then $Sh_c(X;\Lambda)$ (resp. $Sh_{cc}(X;\Lambda)$) denotes the full subcategory of $Sh_c(X)$ (resp. $Sh_{cc}(X)$) whose objects have singular support in Λ.

2.4. **Toric geometry.** Let $N \cong \mathbb{Z}^n$ be a free abelian group, and let Σ be a *fan* in N (or in $N_\mathbb{R} = N \otimes \mathbb{R}$) of strongly convex rational polyhedral cones. We do not necessarily assume that Σ satisfies further conditions—e.g. that it is complete, or simplicial.

[2] A triangle in an A_∞ category C is distinguished if it induces a distinguished triangle in the cohomology category $H(C)$.

[3] Here is a construction of the unique-up-to-isomorphism triangulated envelope. The Yoneda embedding $\mathcal{Y} : C \to mod(C)$ maps an object L of a category C to the A_∞ right C-module $hom_C(-, L)$. The functor $\mathcal{Y}$ is a quasi-embedding of C into the triangulated category $mod(C)$. Then the triangulated completion $Tr(C)$ is the category of twisted complexes of representable modules in $mod(C)$.

2.4.1. *Notation.* Given N and Σ, we fix the following standard notation:

- $M := \mathrm{Hom}(N, \mathbb{Z}) =: N^\vee$ is the dual lattice to N.
- $N_\mathbb{R}$ and $M_\mathbb{R}$ are the real vector spaces spanned by N and M, i.e. $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$.
- X_Σ is the complex toric variety associated to Σ. It is naturally equipped with an action of the algebraic torus $T = N \otimes \mathbb{C}^*$.

We also use:

- $T_\mathbb{R}$ denotes the maximal compact subgroup of T. So $T_\mathbb{R} \cong N_\mathbb{R}/N \cong \mathrm{U}(1)^n$.
- Dually, $T^\vee := M \otimes \mathbb{C}^*$ and $T^\vee_\mathbb{R} \cong M_\mathbb{R}/M$ is its maximal compact.
- $\Sigma(d)$ is the set of d-dimensional cones in Σ. In particular, $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ is the set of rays. Let $v_i \in N$ be the generator of ρ_i, i.e. $\rho_i \cap N = \mathbb{Z}_{\geq 0} v_i$.
- Let $\langle\ ,\ \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ denote the natural pairing.
- Given a cone $\sigma \in \Sigma$, let
$$\sigma^\vee = \{x \in M_\mathbb{R} \mid \langle x, y\rangle \geq 0 \text{ for all } y \in N_\mathbb{R}\}$$
be the dual cone, and define
$$\sigma^\perp = \{x \in M_\mathbb{R} \mid \langle x, y\rangle = 0 \text{ for all } y \in N_\mathbb{R}\}.$$
If σ is a d-dimensional cone then $\sigma^\perp \subset M_\mathbb{R}$ is a codimension-d $\mathbb{R}$-linear subspace.

2.4.2. *Equivariant line bundles.* Let D_i be the $(n-1)$-dimensional T orbit closure associated to ρ_i, so that D_i is a T-divisor of X. Any T-divisor D of X is of the form $D_{\vec{c}} = \sum_{i=1}^r c_i D_i$, where $\vec{c} = (c_1, \dots, c_r) \in \mathbb{Z}^r$, and any T-equivariant line bundle on X is of the form $\mathcal{L}_{\vec{c}} = \mathcal{O}_{X_\Sigma}(D_{\vec{c}})$. If $\mathcal{L}_{\vec{c}}$ is ample then

$$\triangle_{\vec{c}} := \{m \in M_\mathbb{R} \mid \langle m, v_i\rangle \geq -c_i, i = 1, \dots, r\} \tag{5}$$

is a convex polytope in $M_\mathbb{R}$.

2.4.3. *Orbits.* The T-orbits of X_Σ can be described using the structure of the fan. Given a d-dimensional cone $\tau \in \Sigma$, let $\tau^\perp$ be the $(n-d)$-dimensional subspace of $M_\mathbb{R}$ defined by
$$\tau^\perp = \{m \in M_\mathbb{R} \mid \langle m, y\rangle = 0\ \forall y \in \sigma\}.$$
Let N_τ be the rank d sublattice of N generated by $\tau \cap N$, and let
$$N(\tau) = N/N_\tau, \quad M(\tau) = \tau^\perp \cap M.$$
Then $N(\tau)$ and $M(\tau)$ are dual lattices of rank $(n-d)$, and
$$O_\tau = \mathrm{Hom}(M(\tau), \mathbb{C}^*) = \mathrm{Spec}\, \mathbb{C}[M(\tau)] = N(\tau) \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^{n-d}$$
is a T-orbit in X_Σ. The stabilizer of any point in O_τ is $T_\tau := N_\tau \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^d$. In particular, $O_{\{0\}} = \mathrm{Hom}(M, \mathbb{C}^*) = N \otimes \mathbb{C}^* = T \cong (\mathbb{C}^*)^n$.

We have a disjoint union of T-orbits:

$$X_\Sigma = \bigcup_{\tau \in \Sigma} O_\tau, \tag{6}$$

which is a T-equivariant stratification of X_Σ. Let
$$X_\tau = \mathrm{Spec}\, \mathbb{C}[\tau^\vee \cap M] \cong (\mathbb{C}^*)^{n-d} \times \mathbb{C}^d$$

be the affine toric subvariety of X_Σ associated to τ. There is an inclusion $O_\tau \subset X_\tau$ and a deformation retraction $r_\tau : X_\tau \to O_\tau$. More explicitly, there exists a basis $w_1, \ldots, w_d$ of N_τ such that

$$\tau = \{r_1 w_1 + \ldots + r_d w_d \mid r_i \geq 0\}.$$

Define

$$\tau^\circ = \{r_1 w_1 + \ldots + r_d w_d \mid r_i > 0\}.$$

Suppose that $y \in N_\tau \cap \tau^\circ$, so that $y = \sum_{j=1}^d n_j w_j$ where $n_j \in \mathbb{Z}_{>0}$. Then the retraction r_τ is given by

$$r_\tau(p) = \lim_{t \to -\infty} e^{ty} \cdot p.$$

Since $T = O_{\{0\}}$ is contained in X_τ for all $\tau \in \Sigma$, we have a surjective map $r_\tau : O_{\{0\}} \to O_\tau$ which can be identified with the natural projection $T \to T/T_\tau$.

There is an inclusion $j : \mathbb{R}^+ = \{e^y \mid y \in \mathbb{R}\} \hookrightarrow \mathbb{C}^* = \{e^y \mid y \in \mathbb{C}\} = \mathbb{C}^*$ and a retraction $r : \mathbb{C}^* \to \mathbb{R}^+$ given by $z \mapsto |z|$. This induces inclusions

$$j : O_\tau^+ \stackrel{\mathrm{def}}{=} N(\tau) \otimes \mathbb{R}^+ \hookrightarrow O_\tau = N(\tau) \otimes \mathbb{C}^*$$

and retractions

$$r : O_\tau = N(\tau) \otimes \mathbb{C}^* \to O_\tau^+ = N(\tau) \otimes \mathbb{R}^+.$$

In particular, $O_{\{0\}}^+ \cong (\mathbb{R}^+)^n$ is the image of the inclusion $\exp : N_\mathbb{R} \to T$ given by $y \mapsto \exp(y)$. For each $\tau \in \Sigma$, we have a surjective map

$$r_\tau^+ : O_{\{0\}}^+ \cong (\mathbb{R}^+)^n \to O_\tau^+ \cong (\mathbb{R}^+)^{n-d}.$$

Let

$$(X_\Sigma)_{\geq 0} = \bigcup_{\tau \in \Sigma} O_\tau^+. \tag{7}$$

Then we have an inclusion $j : (X_\Sigma)_{\geq 0} \hookrightarrow X_\Sigma$ and a retraction $r : X_\Sigma \to (X_\Sigma)_{\geq 0}$. The retraction r descends to a homeomorphism $X_\Sigma / T_\mathbb{R} \cong (X_\Sigma)_{\geq 0}$.

3. Homological Mirror Symmetry for Toric Varieties

In this section, we derive theorems relating the category of coherent (equivariant) sheaves on X_Σ to the Fukaya category on $T^* T_\mathbb{R}^\vee$ ($T^* M_\mathbb{R}$).

3.1. **The Coherent-Constructible Correspondence.** In this subsection, we briefly recall the results of [FLTZ, Tr]. The results in [FLTZ, Tr] hold for toric varieties over an arbitrary commutative, Noetherian base ring R. Here we state the results for the case $\mathsf{R} = \mathbb{C}$. We use the notation in Section 2.

Let X_Σ be a toric variety defined by a *complete* fan $\Sigma \subset N_\mathbb{R}$. We define

$$\Lambda_\Sigma = \bigcup_{\tau \in \Sigma} (\tau^\perp + M) \times -\tau \subset M_\mathbb{R} \times N_\mathbb{R} = T^* M_\mathbb{R}. \tag{8}$$

where $\tau^\perp + M = \{x + \chi \mid x \in \tau^\perp, \chi \in M\}$. Then Λ_Σ is a Lagrangian subvariety of $T^* M_\mathbb{R}$. Let $\bar{\Lambda}_\Sigma \subset T^* T_\mathbb{R}^\vee$ be the image of Λ_Σ under the universal covering map $T^* M_\mathbb{R} = M_\mathbb{R} \times N_\mathbb{R} \to T^* T_\mathbb{R}^\vee = T_\mathbb{R}^\vee \times N_\mathbb{R}$.

Theorem 3.1 (equivariant coherent-constructible correspondence [FLTZ])**.** *Let X_Σ be a complete toric variety defined by a finite complete fan $\Sigma \subset N_\mathbb{R}$. Then there is a quasi-equivalence of monoidal dg categories*

$$\kappa : \mathcal{P}\mathrm{erf}_T(X_\Sigma) \longrightarrow Sh_{cc}(M_\mathbb{R}; \Lambda_\Sigma). \tag{9}$$

The functor κ sends an equivariant ample line bundle $\mathcal{L}_{\vec{c}}$ on X_Σ to the costandard constructible sheaf $i_! \omega_{\triangle^\circ_{\vec{c}}}$ on $M_\mathbb{R}$, where $\triangle^\circ_{\vec{c}}$ is the interior of the convex polytope $\triangle_{\vec{c}}$.

Theorem 3.2 (nonequivariant coherent-constructible correspondence [Tr])**.** *There is a quasi-embedding of monoidal dg categories:*

$$\bar{\kappa} : \mathcal{P}\mathrm{erf}(X_\Sigma) \longrightarrow Sh_c(T^\vee_\mathbb{R}; \bar{\Lambda}_\Sigma) \tag{10}$$

which makes the following square commute up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{P}\mathrm{erf}_T(X_\Sigma) & \xrightarrow{\kappa} & Sh_{cc}(M_\mathbb{R}; \Lambda_\Sigma) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p_!} \\ \mathcal{P}\mathrm{erf}(X_\Sigma) & \xrightarrow{\bar{\kappa}} & Sh_c(T^\vee_\mathbb{R}; \bar{\Lambda}_\Sigma) \end{array} \tag{11}$$

where f forgets the equivariant structure, and $p : M_\mathbb{R} \to T^\vee_\mathbb{R} = M_\mathbb{R}/M$ is the natural projection.

Remarks 3.3.
(1) The monoidal structures in Theorem 3.1 and Theorem 3.2 will be discussed in Section 3.4 below.
(2) In [FLTZ], κ is defined in terms of certain equivariant quasicoherent sheaves that arise naturally in the Čech resolution.
(3) In [Tr], the third author proved that (10) is a quasi-equivalence when X_Σ is a projective, unimodular, zonotopal toric variety. We conjecture that (10) is a quasi-equivalence for any complete toric variety.

3.2. **The unwrapped Fukaya category.** The Fukaya category of the cotangent T^*X of a compact real analytic manifold X was defined in [NZ] and equated with constructible sheaves on X in [NZ, N1]. Here we review aspects most relevant to the present case, including the role of infinity and of standard branes.

Let X be a real analytic manifold equipped with a Riemannian metric g. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Define the closed unit disc bundle to be

$$D^*X = \{(x, \xi) \in T^*X \mid \|\xi\| \leq 1\} \subset T^*X,$$

and define the unit sphere bundle to be

$$S^*X = \{(x, \xi) \in T^*X \mid \|\xi\| = 1\} = \partial(D^*X) \subset T^*X.$$

We may think of D^*X as a compactification $\overline{T}^*X$ of T^*X by the following compactification map

$$\begin{array}{rccc} \iota : & T^*X & \to & D^*X \\ & (x,\xi) & \mapsto & \Big(x, \dfrac{\xi}{\sqrt{1+\|\xi\|^2}}\Big). \end{array} \tag{12}$$

and we can think of S^*X as $T^\infty X$ because it is the "infinity" part of T^*X under this compactification. If L is a Lagrangian submanifold of T^*X we write L^∞ for $\overline{\iota(L)} \cap S^*X$, the part of L at infinity in the fibers.

In the present case, $M_{\mathbb{R}}$ is noncompact. This is only a minor complication, as we will require all Lagrangian branes L to have compact *horizontal support*, i.e., $\overline{\pi(L)}$ is compact. Define the flat metric g on $T^*M_{\mathbb{R}} = M_{\mathbb{R}} \times N_{\mathbb{R}}$ by declaring a $\mathbb{Z}$ basis $\{e_1, \ldots, e_n\}$ of $N \subset N_{\mathbb{R}}$ to be orthonormal, and likewise for the dual basis $\{e_1^*, \ldots, e_n^*\}$ of M. Then $\overline{\pi(L)}$ is bounded. We require as well that the usual other conditions of Lagrangian branes are satisfied: that is, L must be an exact Lagrangian submanifold of $T^*M_{\mathbb{R}}$; $\overline{\iota(L)}$ is a $\mathcal{C}$-set of D^*X;[4] and L is equipped with the data of a vector bundle with flat connection, a brane structure and a tame perturbation (see [NZ]). Under these conditions, morphisms are well-defined for the following reason. If $L = (L_1, ..., L_k)$ is a finite collection of Lagrangian objects with compact horizontal support, then there exists a sublattice $\Xi \subset M$ of finite index d such that the union of the supports of the L_i are contained in a single fundamental domain: then all morphisms can be computed in the cotangent of the compact torus $M_{\mathbb{R}}/\Xi$ (a degree d cover of the torus $T^\vee_{\mathbb{R}} = M_{\mathbb{R}}/M$) and lifted to $T^*M_{\mathbb{R}} = N_{\mathbb{R}} \times M_{\mathbb{R}}$—see [N1, Section 5.3] for details.[5] Holomorphicity is preserved by the lift since the quotient by Ξ is a local isomorphism of the Kähler structure. The triangulated envelope of the Fukaya A_∞-category of all such branes is denoted by $Fuk(T^*M_{\mathbb{R}})$.[6]

Let $\Lambda \subset T^*X$ be a conical Lagrangian subset. The A_∞-category generated by Lagrangian branes L with $L^\infty \subset \Lambda^\infty$ is denoted by $Fuk(T^*X; \Lambda)$. Here we will mainly be concerned with $Fuk(T^*M_{\mathbb{R}}; \Lambda_\Sigma)$, where Λ_Σ is given in (8).

3.3. **Microlocalization.** Recall that if $i : Y \hookrightarrow X$ is the inclusion of an analytic submanifold in a compact, real analytic manifold X then $i_*\mathbb{C}_Y$ is the standard object in $Sh_c(X)$ associated to Y, and under microlocalization, the *standard brane* $\mu(i_*\mathbb{C}_Y)$ is defined by the *standard Lagrangian* $L_{Y,*} \subset T^*X$ given by the fiberwise sum

$$L_{Y,*} = T^*_Y X + \Gamma_{df},$$

where $f = \log m$ and m is a nonnegative $\mathcal{C}$-function $m : X \to \mathbb{R}$ that vanishes precisely on the boundary $\partial Y \subset X$. Here $T^*_Y X$ is the conormal bundle of Y in X, and $\Gamma_{df} \subset T^*Y \cong T^*X/T^*_Y X$ is the graph of df. There is a canonical brane structure on this Lagrangian (Section 5.3 of [NZ]). We let $L_{Y,m,*}$ denote the standard Lagrangian defined by a particular choice of m. Two different choices m_1, m_2 give rise to isomorphic objects: $L_{Y,m_1,*} \cong L_{Y,m_2,*}$ as objects in $Fuk(T^*X)$.

Let α be a diffeomorphism on $M_{\mathbb{R}} \times N_{\mathbb{R}}$ given by $\alpha(x, y) = (x, -y)$. A *costandard brane* (*costandard Lagrangian*) L is a brane (Lagrangian) such that $\alpha(L)$ is a standard brane (Lagrangian). Microlocalization μ also takes the costandard constructible sheaf $i_!\omega_Y$ to the costandard brane $L_{Y,!} := T^*_Y X - \Gamma_{df}$. We summarize these results as a theorem.

Theorem 3.4 ([NZ, N1])**.** *There is a quasi-equivalence of A_∞-categories*

$$\mu : Sh_{cc}(M_{\mathbb{R}}; \Lambda_\Sigma) \to Fuk(T^*M_{\mathbb{R}}; \Lambda_\Sigma).$$

---

[4]See Section A.1 for a brief review of analytic-geometric categories, including definitions of $\mathcal{C}$-sets and $\mathcal{C}$-maps.

[5]The condition of compact horizontal support can be dropped for a single given object, as one can define the Yoneda image by analyzing hom's against objects with compact horizontal support—see [N2].

[6]The triangulated envelope of any A_∞-category is unique up to an exact quasi-equivalence.

For any analytic submanifold $Y \subset M_{\mathbb{R}}$, μ takes the standard constructible sheaf $i_*\mathbb{C}_Y$ to the standard brane $L_{Y,*}$ of Y, and takes the costandard constructible sheaf $i_!(\omega_Y)$ to the costandard brane $L_{Y,!}$. The functor μ admits a quasi-inverse μ^{-1} : $Fuk(T^*M_{\mathbb{R}};\Lambda_\Sigma) \to Sh_{cc}(M_{\mathbb{R}};\Lambda_\Sigma)$.

Similarly, we have a quasi-equivalence of A_∞-categories

$$\bar{\mu} : Sh_c(T^\vee_{\mathbb{R}};\bar{\Lambda}_\Sigma) \xrightarrow{\cong} Fuk(T^*T^\vee_{\mathbb{R}};\bar{\Lambda}_\Sigma). \tag{13}$$

3.4. **Functoriality and monoidal structure.** The functoriality of the functor κ is proven in [FLTZ]. We extend this functoriality involving the Fukaya category. This is simply a combination of the results of [FLTZ] and of [N1, Section 5].

We first review some general results in [N1, Section 5]. Given two real analytic manifolds X_0, X_1, let $p_0 : X_0 \times X_1 \to X_0$ and $p_1 : X_0 \times X_1 \to X_1$ be projections. For a real analytic manifold Y, let $\mu_Y : Sh_c(Y) \to Fuk(T^*Y)$ be the microlocalization functor, and let $\alpha_Y : Fuk(T^*Y) \to Fuk(T^*Y)^\circ$ be the brane duality functor (induced by multiplication by -1 on cotangent vectors). Let

$$\mathcal{Y}_\ell : Fuk(T^*X_1) \to mod_l(T^*X_1)^\circ, \quad P \mapsto hom_{Fuk(T^*X_1)}(P,-)$$

be the Yoneda embedding for left A^∞-modules over $Fuk(T^*X_1)$.

An object $\mathcal{K}$ of $Sh_c(X_0 \times X_1)$ defines a functor

$$\Phi_{\mathcal{K}!} : Sh_c(X_0) \to Sh_c(X_1), \quad \mathcal{F} \mapsto p_{1!}(\mathcal{K} \otimes p_0^*\mathcal{F}). \tag{14}$$

An object L of $Fuk(T^*X_0 \times T^*X_1)$ defines a functor

$$\begin{aligned} \tilde{\Psi}_{L!} :& Fuk(T^*X_0) \to mod_\ell(Fuk(T^*X_1))^\circ, \\ & P \mapsto hom_{Fuk(T^*X_0 \times T^*X_1)}(L, \alpha_{X_0}(P) \times -) \end{aligned} \tag{15}$$

The following is a special case of [N1, Proposition 5.3.1].

Theorem 3.5. *Consider an object $\mathcal{K}$ of $Sh_c(X_0 \times X_1)$, and its microlocalization $L = \mu_{X_0 \times X_1}(\mathcal{K})$. Then there is a quasi-isomorphism*

$$\mathcal{Y}_\ell \circ \mu_{X_1} \circ \Phi_{\mathcal{K}!} \simeq \tilde{\Psi}_{L!} \circ \mu_{X_0}.$$

Therefore the functor $\tilde{\Psi}_{L!}$ is represented by

$$\Psi_{L!} := \mu_{X_1} \circ \Phi_{\mathcal{K}!} \circ \mu_{X_0}^{-1} : Fuk(T^*X_0) \to Fuk(T^*X_1).$$

Example 3.6. Let $v : X_0 \to X_1$ be a smooth map, and let

$$\Gamma_v = \{(x_0, x_1) \in X_0 \times X_1 \mid x_1 = v(x_0)\}$$

be the graph of v. Let $\mathcal{K} = \mathbb{C}_{\Gamma_v}$ be the constant sheaf on Γ_v, and let $L_v = \mu_{X_1 \times X_2}(\mathcal{K})$. Then

$$\Phi_{\mathcal{K}!} = v_!, \quad L_v \simeq T^*_{\Gamma_v}(X_0 \times X_1)$$

where $T^*_{\Gamma_v}(X_0 \times X_1)$ is the conormal bundle of Γ_v in $X_0 \times X_1$. Define $\Psi_{L_v!}$ as in Theorem 3.5. Then there is a quasi-isomorphism

$$\Psi_{L_v!} \circ \mu_{X_0} \simeq \mu_{X_1} \circ v_!. \tag{16}$$

For two toric varieties $X_1 = X_{\Sigma_1}$ and $X_2 = X_{\Sigma_2}$ and a fan-preserving map $f : N_1 \to N_2$, let $v : M_{2,\mathbb{R}} \to M_{1,\mathbb{R}}$ and $u : X_1 \to X_2$ be the induced map of vector spaces and varieties (see [FLTZ]). As a special case of Example 3.6, define $L_v := T^*_{\Gamma_v}(M_{2,\mathbb{R}} \times M_{1,\mathbb{R}})$, which is a Lagrangian subspace of $T^*M_{2,\mathbb{R}} \times T^*M_{1,\mathbb{R}} = M_{\mathbb{R},2} \times N_{\mathbb{R},2} \times M_{\mathbb{R},1} \times N_{\mathbb{R},1}$. Combining (16) with Theorem 3.5 and results in [FLTZ, Section 3], we come to a larger diagram:

Theorem 3.7. *For two complete toric varieties $X_1 = X_{\Sigma_1}$ and $X_2 = X_{\Sigma_2}$ and a fan-preserving map $f : N_1 \to N_2$, where f is injective, and associated maps $f \otimes 1_{\mathbb{C}^*} : T_1 \to T_2$, $u : X_1 \to X_2$, $v : M_{2,\mathbb{R}} \to M_{1,\mathbb{R}}$, the following diagram commutes up to a quasi-isomorphism.*

$$\begin{tikzcd}
\mathcal{P}\mathrm{erf}_{T_2}(X_2) \arrow[r, "\kappa_2"] \arrow[d, "u^*"'] & Sh_{cc}(M_{2,\mathbb{R}};\Lambda_{\Sigma_2}) \arrow[r, "\mu_{M_{2,\mathbb{R}}}"] \arrow[d, "v_!"] & Fuk(T^*M_{2,\mathbb{R}};\Lambda_{\Sigma_2}) \arrow[d, "\Psi_{L_v!}"] \\
\mathcal{P}\mathrm{erf}_{T_1}(X_1) \arrow[r, "\kappa_1"'] & Sh_{cc}(M_{1,\mathbb{R}};\Lambda_{\Sigma_1}) \arrow[r, "\mu_{M_{1,\mathbb{R}}}"'] & Fuk(T^*M_{1,\mathbb{R}};\Lambda_{\Sigma_1}).
\end{tikzcd}$$

Example 3.8 (a product structure on the Fukaya category)**.** This example is a special case of Example 3.6.

Let G be a Lie group, and let $v : G \times G \to G$ be the multiplication: $v(g_1, g_2) = g_1 \cdot g_2$. Then L_v is an object in $Fuk(T^*(G \times G) \times T^*G)$ and defines a functor $\Psi_{L_v!} : Fuk(T^*G \times T^*G) \to Fuk(T^*G)$. We define the product $L_1 \diamond L_2$ of two objects L_1 and L_2 of $Fuk(T^*G)$ by the formula

$$L_1 \diamond L_2 := \Psi_{L_v!}(L_1 \times L_2). \tag{17}$$

Proposition 3.9 (the microlocalization intertwines the product structures)**.** *Let G be a Lie group. The microlocalization functor $\mu_G : Sh_{cc}(G) \xrightarrow{\sim} Fuk(T^*G)$ intertwines the monoidal product on $Sh_{cc}(G)$ given by the convolution, and the product structure on $Fuk(T^*G)$ given by the product $\diamond$ defined by (17), up to a quasi-isomorphism: i.e. the functors $\mu_G(- \star -)$ and $\mu_G(-) \diamond \mu_G(-)$ are quasi-isomorphic in the category of A_∞-functors from $Sh_{cc}(G) \times Sh_{cc}(G)$ to $Fuk(T^*G)$.*

Proof. Recall that convolution product $F_1 \star F_2$ of two objects F_1 and F_2 of $Sh_{cc}(G)$ is defined by $F_1 \star F_2 = v_!(F_1 \boxtimes F_2)$. So

$$\begin{aligned}
\mu_G(F_1 \star F_2) &= \mu_G \circ v_!(F_1 \boxtimes F_2) \cong \Psi_{L_v!} \circ \mu_{G\times G}(F_1 \boxtimes F_2) \\
&= \Psi_{L_v!}(\mu_G(F_1) \times \mu_G(F_2)) = \mu_G(F_1) \diamond \mu_G(F_2)
\end{aligned}$$

□

3.5. **Equivariant and nonequivariant HMS for toric varieties.** Let $\tau = \mu \circ \kappa$ and let $\bar{\tau} = \bar{\mu} \circ \bar{\kappa}$. Notice that the convolution product of costandard sheaves $i_{1!}\omega_{\Delta_{\vec{c}_1}}$ and $i_{2!}\omega_{\Delta_{\vec{c}_2}}$ is $i_!\omega_{\Delta_{\vec{c}_1+\vec{c}_2}}$, where c_1 and c_2 determine two equivariant ample line bundles on X_Σ, and i_1, i_2 and i are corresponding embeddings of polytopes. Since costandard sheaves over convex polytopes of ample line bundles generate the category $Sh_{cc}(M_\mathbb{R};\Lambda_\Sigma)$, as shown in [FLTZ], the subcategory $Sh_{cc}(M_\mathbb{R};\Lambda_\Sigma)$ of $Sh_{cc}(M_\mathbb{R};\Lambda_\Sigma)$ is closed under the convolution product. By results in [Tr], the subcategory $Sh_c(T^\vee_\mathbb{R}, \bar{\Lambda}_\Sigma)$ of $Sh_c(T^\vee_\mathbb{R})$ is closed under the convolution product. Combining Theorem 3.1, Theorem 3.2, Theorem 3.4, and Proposition 3.9, we obtain:

Theorem 3.10. *Let X_Σ be a complete toric variety defined by a finite complete fan $\Sigma \subset N_\mathbb{R}$. Then there is an quasi-equivalence of A_∞ categories*

$$\tau : \mathcal{P}\mathrm{erf}_T(X_\Sigma) \xrightarrow{\cong} Fuk(T^*M_\mathbb{R};\Lambda_\Sigma). \tag{18}$$

There is an quasi-embedding of A_∞ categories

$$\bar{\tau} : \mathcal{P}\mathrm{erf}(X_\Sigma) \to Fuk(T^*T^\vee_\mathbb{R};\bar{\Lambda}_\Sigma). \tag{19}$$

The functors τ and $\bar{\tau}$ intertwine the corresponding product structures in the Fukaya categories and the monoidal products in the dg category of perfect sheaves. More precisely, there are quasi-isomorphisms

$$\tau(-\otimes-)\cong\tau(-)\diamond\tau(-),\ \ \bar{\tau}(-\otimes-)\cong\bar{\tau}(-)\diamond\bar{\tau}(-).$$

Let $DCoh_T(X_\Sigma)$ be the bounded derived category of T-equivariant coherent sheaves on X_Σ, and let $DCoh(X_\Sigma)$ be the bounded derived category of coherent sheaves on X_Σ. When X_Σ is nonsingular, we have $D\mathcal{P}\mathrm{erf}_T(X_\Sigma)=DCoh_T(X_\Sigma)$ and $D\mathcal{P}\mathrm{erf}(X_\Sigma)=DCoh(X_\Sigma)$. Taking H^0 of (18) and (19), we obtain the following Corollary 3.11 and Corollary 3.12, respectively.

Corollary 3.11 (Equivariant homological mirror symmetry of toric varieties)**.** *Let X_Σ be a nonsingular complete toric variety defined by a finite nonsingular complete fan $\Sigma\subset N_\mathbb{R}$. Then there is an equivalence of tensor triangulated categories*

$$H(\tau): DCoh_T(X_\Sigma)\xrightarrow{\cong} DFuk(T^*M_\mathbb{R};\Lambda_\Sigma) \tag{20}$$

Corollary 3.12 (Homological mirror symmetry of toric varieties)**.** *Let X_Σ be a nonsingular complete toric variety defined by a finite nonsingular complete fan $\Sigma\subset N_\mathbb{R}$. There is an embedding of tensor triangulated categories*

$$H(\bar{\tau}): DCoh(X_\Sigma)\longrightarrow DFuk(T^*T^\vee_\mathbb{R};\bar{\Lambda}_\Sigma). \tag{21}$$

We conjecture that (21) is an equivalence. This is proven for Σ a complete, unimodular hyperplane arrangement in [Tr].

4. T-duality

In this section, we perform an equivariant version of T-duality. Let X_Σ be an n-dimensional nonsingular projective toric variety (so that it is a compact toric manifold). Then $T\cong(\mathbb{C}^*)^n$ and its maximal compact subgroup $T_\mathbb{R}\cong U(1)^n$ acts on X_Σ. From a T-equivariant line bundle $\mathcal{L}_{\vec{c}}$ on X together with a $T_\mathbb{R}$-invariant hermitian metric h, we construct a Lagrangian submanifold $\mathbb{L}_{\vec{c},h}$ of $T^*M_\mathbb{R}$. We relate $\mathbb{L}_{\vec{c},h}$ to classical objects in symplectic geometry.

4.1. **Construction of the T-dual Lagrangian.** Let $X=X_\Sigma$ be a smooth projective toric variety defined by a fan $\Sigma\subset N_\mathbb{R}$, and let $\rho_1,\ldots,\rho_r$ be the 1-dimensional cones in Σ and $D_1,\ldots,D_r$ the associated T-divisors, as in Section 2.4.

There exists $s_i\in H^0(X,\mathcal{O}_X(D_i))$, unique up to multiplication by a constant scalar in $\mathbb{C}^*$, such that the zero locus of s_i is exactly D_i.

$$X_{\{0\}}=X\setminus\cup_{i=1}^r D_i=\mathrm{Spec}\mathbb{C}[M]\cong(\mathbb{C}^*)^n.$$

is the unique open orbit of the T-action.

The meromorphic section $s_{\vec{c}}:=\prod_{i=1}^r s_i^{c_i}$ of $\mathcal{L}_{\vec{c}}=\mathcal{O}_X(D_{\vec{c}})$ is defined up to multiplication by a constant scalar in $\mathbb{C}^*$. The restriction of $s_{\vec{c}}$ to $X_{\{0\}}$ is a holomorphic frame of $\mathcal{L}_{\vec{c}}$ on the Zariski open subset $X_{\{0\}}\subset X$.

We now choose a $T_\mathbb{R}$-invariant, real analytic hermitian metric h on $\mathcal{L}_{\vec{c}}$. Let $\nabla_{\vec{c},h}$ be the unique connection on $\mathcal{L}_{\vec{c}}$ determined by the holomorphic structure on $\mathcal{L}_{\vec{c}}$ and the hermitian metric h. The connection 1-form of ∇_h with respect to the unitary frame $s_{\vec{c}}/\|s_{\vec{c}}\|_h$ of $\mathcal{L}_{\vec{c}}|_{X_{\{0\}}}$ is the following purely imaginary, real analytic 1-form.

$$\alpha=-2\sqrt{-1}\mathrm{Im}(\bar{\partial}\log\|s_{\vec{c}}\|_h).$$

Note that α is invariant if we replace $s_{\vec{c}}$ and h by $\lambda s_{\vec{c}}$ and ρh respectively, where $\lambda \in \mathbb{C}^*$ and $\rho \in (0,\infty)$ are constants.

We now introduce coordinates on $X_{\{0\}} \cong T$ (the identification depends on the choice of a point in $X_{\{0\}}$). The universal cover of T can be canonically identified with $N \otimes \mathbb{C} = N_{\mathbb{R}} \times N_{\mathbb{R}}$. Let $\{e_1, \dots, e_n\}$ be a $\mathbb{Z}$-basis of the lattice N, and let $\{e_1^*, \dots, e_n^*\}$ be a dual $\mathbb{Z}$-basis of the dual lattice M. A vector in $N \otimes \mathbb{C}$ is of the form $\sum_{j=1}^n \frac{y_j + \sqrt{-1}\theta_j}{2\pi} e_j$ where $y_j, \theta_j \in \mathbb{R}$. A vector in $M_{\mathbb{R}}$ is of the form $\sum_{j=1}^n \frac{\gamma_j}{2\pi} e_j^*$, where $\gamma_j \in \mathbb{R}$. Then $y_j + \sqrt{-1}\theta_j$ are complex coordinates on $N_{\mathbb{C}}$, and γ_j, y_j are Darboux coordinates on $T^* M_{\mathbb{R}} = M_{\mathbb{R}} \times N_{\mathbb{R}}$. The symplectic form on $M_{\mathbb{R}} \times N_{\mathbb{R}}$ is

$$\omega^\vee = \sum_{j=1}^n dy_j \wedge d\gamma_i$$

which descends to a symplectic form on $(M_{\mathbb{R}}/M) \times N_{\mathbb{R}} \cong T^*(T_{\mathbb{R}}^\vee)$. Note that $M \subset M_{\mathbb{R}}$ is given by $\gamma_j \in 2\pi\mathbb{Z}$ and $N \subset N_{\mathbb{R}}$ is given by $\theta_j \in 2\pi\mathbb{Z}$. Let $r_j = e^{y_j}$, so that the coordinates on T are $e^{y_j + \sqrt{-1}\theta_j} = r_j e^{\sqrt{-1}\theta_j}$, $j = 1, \dots, n$.

The function $\|s\|_h$ is $T_{\mathbb{R}}$-invariant, so it depends on r_i (y_i) but not on θ_i. We have

$$\begin{aligned} \sqrt{-1}\alpha &= 2\mathrm{Im}(\bar\partial \log \|s_{\vec{c}}\|_h) = \mathrm{Im}\Big(\sum_{j=1}^n \big(\frac{\partial}{\partial r_j} \log \|s_{\vec{c}}\|_h\big) \cdot (dr_j - \sqrt{-1} r_j d\theta_j)\Big) \\ &= -\sum_{j=1}^n \big(\frac{\partial}{\partial y_j} \log \|s_{\vec{c}}\|_h\big) d\theta_j. \end{aligned}$$

Let $y = (y_1, \dots, y_n)$, and let $f_{\vec{c},h}(y) = -\log \|s_{\vec{c}}\|_h$. Then $f_{\vec{c},h}(y)$ is a real analytic function in y, and

$$\sqrt{-1}\alpha = \sum_{j=1}^n \frac{\partial f_{\vec{c},h}}{\partial y_j}(y) d\theta_j. \tag{22}$$

We now T-dualize following [AP]. Specifically, the data of a Lagrangian section of the dual torus fibration $T_{\mathbb{R}}^\vee \times N_{\mathbb{R}} \to N_{\mathbb{R}}$ (projection to the second factor) is equated with a $T_{\mathbb{R}}$-invariant $U(1)$-connection on the torus fibration $p_2 : T_{\mathbb{R}} \times N_{\mathbb{R}} \to N_{\mathbb{R}}$ (projection to the second factor). The restriction of α to a fiber $p_2^{-1}(y) \cong T_{\mathbb{R}}$ is a harmonic 1-form on the torus $p_2^{-1}(y)$, which can be viewed as an element in $H^1(T_{\mathbb{R}}; \mathbb{R}) \cong M_{\mathbb{R}}$, the universal cover of the dual torus $T_{\mathbb{R}}^\vee = M_{\mathbb{R}}/M$ of $T_{\mathbb{R}}$. Let $\mathbb{L}_{\vec{c},h} \subset M_{\mathbb{R}} \times N_{\mathbb{R}}$ be the graph of the map $N_{\mathbb{R}} \to M_{\mathbb{R}}$ defined by $y \mapsto \sqrt{-1}\alpha\Big|_{p_2^{-1}(y)}$. In terms of the coordinates γ_j on $M_{\mathbb{R}}$ and y_j on $N_{\mathbb{R}}$, $\mathbb{L}_{\vec{c},h}$ is given by

$$\frac{\gamma_j}{2\pi} = \frac{\partial f_{\vec{c},h}}{\partial y_j}(y), \quad j = 1, \dots, n.$$

Since $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ are dual real vector spaces, we have $M_{\mathbb{R}} \times N_{\mathbb{R}} \cong T^* N_{\mathbb{R}} \cong T^* M_{\mathbb{R}}$. Moreover, the canonical symplectic forms on $T^* N_{\mathbb{R}}$ and $T^* M_{\mathbb{R}}$ are

$$-\omega^\vee = \sum_{j=1}^n d\gamma_j \wedge dy_j, \quad \omega^\vee = \sum_{j=1}^n dy_j \wedge d\gamma_j.$$

f_h is a real analytic function on $N_{\mathbb{R}}$. The submanifold $\mathbb{L}_{\vec{c},h}$ is the graph of df_h in $T^* N_{\mathbb{R}}$, so it is a real analytic Lagrangian submanifold of $(T^* N_{\mathbb{R}}, -\omega^\vee)$ and of $(T^* M_{\mathbb{R}}, \omega^\vee)$. Let $\bar{\mathbb{L}}_{\vec{c},h} \subset T^* T_{\mathbb{R}}^\vee = (M_{\mathbb{R}}/M) \times N_{\mathbb{R}}$ be the image of $\mathbb{L}_{\vec{c},h}$ under the

projection $M_{\mathbb{R}} \times N_{\mathbb{R}} \to (M_{\mathbb{R}}/M) \times N_{\mathbb{R}}$. Then $\bar{\mathbb{L}}_{\vec{c},h}$ is a real analytic Lagrangian submanifold in $T^*T^\vee_{\mathbb{R}} = T^\vee_{\mathbb{R}} \times N_{\mathbb{R}}$, and is the graph of a map $N_{\mathbb{R}} \to T^\vee_{\mathbb{R}}$. Both $\mathbb{L}_{\vec{c},h}$ and $\bar{\mathbb{L}}_{\vec{c},h}$ are diffeomorphic to $N_{\mathbb{R}} \cong \mathbb{R}^n$, so they are exact Lagrangian submanifolds.

Suppose that $D_{\vec{c}'} - D_{\vec{c}}$ is a principal divisor. Then $\mathcal{L}_{\vec{c}'}$ and $\mathcal{L}_{\vec{c}}$ are the same holomorphic line bundle equipped with possibly different T-equivariant structures, so we may choose the same hermitian metric h on $\mathcal{L}_{\vec{c}'}$ and on $\mathcal{L}_{\vec{c}}$. We have $s_{\vec{c}'} = s_{\vec{c}} \prod_{j=1}^n t_j^{m_j}$ for some $(m_1, \ldots, m_n) \in \mathbb{Z}^n$, so

$$f_{\vec{c}',h} = f_{\vec{c},h} - \sum_{j=1}^n m_j y_j, \quad \frac{\partial f_{\vec{c}',h}}{\partial y_j}(y) = \frac{\partial f_{\vec{c},h}}{\partial y_j}(y) - m_j.$$

Therefore $\bar{\mathbb{L}}_{\vec{c}',h} = \bar{\mathbb{L}}_{\vec{c},h}$.

4.2. **Relations with the equivariant first Chern form and the moment map.**

4.2.1. *Equivariantly closed 2-forms and moment maps of presymplectic forms.* We recall some definitions from [AB] and [KT].

The real vector space $N_{\mathbb{R}}$ can be identified with the Lie algebra of the compact torus $T_{\mathbb{R}}$, with a basis $\{e_1, \ldots, e_n\}$, and $M_{\mathbb{R}}$ is the dual real vector space, with the dual basis $\{e_1^*, \ldots, e_n^*\}$. Let X_j be the vector field on X associated to $e_j \in N_{\mathbb{R}}$. An equivariant 2-form on X is of the form

$$\omega^\# = \omega + \sum_{j=1}^n \phi_j e_j^*$$

where ω is a $T_{\mathbb{R}}$-invariant 2-form on X and ϕ_j are $T_{\mathbb{R}}$-invariant functions on X. An equivariant 2-form $\omega^\#$ is *equivariantly closed* if

$$d\omega = 0 \tag{23}$$

and

$$i_{X_j}\omega + d\phi_j = 0, \quad j = 1, \ldots, n. \tag{24}$$

In this case, the closed 2-form ω represents a cohomology class $[\omega] \in H^2(X; \mathbb{R})$, and the equivariantly closed 2-form represents an equivariant cohomology class $[\omega^\#] \in H^2_T(X; \mathbb{R})$, and we say $\omega^\#$ (resp. $[\omega^\#]$) is an equivariant lifting of ω (resp. $[\omega]$).

In the terminology of [KT], (23) says that ω is a presymplectic form (which is by definition a $T_{\mathbb{R}}$-invariant closed 2-form), and (24) says that $\Phi = \sum_{j=1}^n \phi_j e_j^* : X \to M_{\mathbb{R}}$ is a moment map of the $T_{\mathbb{R}}$-action with respect to the presymplectic form ω. When ω is nondegenerate, ω is a symplectic form, and Φ is a moment map of the $T_{\mathbb{R}}$-action on the symplectic manifold (X, ω).

4.2.2. *The Equivariant first Chern form.* We now return to the construction in Section 4.1. Let F_h be the curvature 2-form of the connection ∇_h. Then

$$c_1(\mathcal{L}_{\vec{c}}, \nabla_h) = \frac{\sqrt{-1}}{2\pi} F_h$$

is a closed, real, $T_{\mathbb{R}}$-invariant, real analytic 2-form which represents the first Chern class $c_1(\mathcal{L}_{\vec{c}}) \in H^2(X; \mathbb{R})$. The closed 2-form $c_1(\mathcal{L}_{\vec{c}}, \nabla_h)$ is known as the first Chern *form* defined by the connection ∇_h; it depends on the underlying holomorphic line bundle and the hermitian metric h, but not on the equivariant structure.

The section $s_{\vec{c}}$ determines an equivariant lifting $c_1(\mathcal{L}_{\vec{c}}, \nabla_h, s_{\vec{c}})$ of the first Chern form $c_1(\mathcal{L}_{\vec{c}}, \nabla_h)$. More explicitly,

$$c_1(\mathcal{L}_{\vec{c}}, \nabla_h, s_{\vec{c}}) = \frac{1}{2\pi}(\sqrt{-1}F_h + \sum_{i=1}^n \phi_j e_j^*)$$

where $\phi_1, \ldots, \phi_n$ are $T_{\mathbb{R}}$-invariant, real-valued functions on X. On the open set $X_{\{0\}} \cong (\mathbb{C}^*)^n$, we have $X_j = \frac{\partial}{\partial \theta_j}$, and

$$\sqrt{-1}F_h = \sqrt{-1}d\alpha = \sum_{j=1}^n d(\frac{\partial f_{\vec{c},h}}{\partial y_j}) \wedge d\theta_j, \qquad \phi_j = \frac{\partial f_{\vec{c},h}}{\partial y_j}(y).$$

The equivariantly closed 2-form $c_1(\mathcal{L}_{\vec{c}}, \nabla_h, s_{\vec{c}})$ represents the equivariant first Chern class $(c_1)_T(\mathcal{L}_{\vec{c}}) \in H_T^2(X; \mathbb{R})$; we call $c_1(\mathcal{L}_{\vec{c}}, \nabla_h, s_{\vec{c}})$ the equivariant first Chern *form* defined by ∇_h and $s_{\vec{c}}$.

4.2.3. *The Moment Map.* The real analytic map $\Phi_{\vec{c},h} = \sum_{j=1}^n \phi_j e_j^* : X \to M_{\mathbb{R}}$ is a moment map of the presymplectic form $\omega_h := \sqrt{-1}F_h$. On $X_{\{0\}}$ it is given by

$$\Phi_{\vec{c},h}(y, \theta) = \sum_{j=1}^n \frac{\partial f_{\vec{c},h}}{\partial y_j}(y) e_j^*.$$

Define new coordinates $x_j = \frac{\gamma_j}{2\pi}$ on $M_{\mathbb{R}}$, so that $M \subset M_{\mathbb{R}}$ is given by $x_j \in \mathbb{Z}$. Then the T-dual Lagrangian $\mathbb{L}_{\vec{c},h}$ constructed in Section 4.1 can be written as

$$\mathbb{L}_{\vec{c},h} = \{(x, y) \in M_{\mathbb{R}} \times N_{\mathbb{R}} \mid x = \Phi_{\vec{c},h} \circ j_0(y)\}$$

where $j_0 : N_{\mathbb{R}} \to X_\Sigma$ is a composition of inclusions:

$$N_{\mathbb{R}} \overset{\exp}{\cong} N \otimes \mathbb{R}^+ \cong (\mathbb{R}^+)^n \hookrightarrow (\mathbb{C}^*)^n \cong N \otimes \mathbb{C}^* = T = X_\Sigma - \cup_{i=1}^r D_i \hookrightarrow X_\Sigma.$$

We also have

$$\Phi_{\vec{c},h} \circ j_0(N_{\mathbb{R}}) = \Phi_h(X_\Sigma - \cup_{i=1}^r D_i).$$

The image of $\Phi_{\vec{c},h} : X \to M_{\mathbb{R}}$ is a *twisted polytope* in the sense of [KT].

4.3. **T-dual Lagrangians of ample and anti-ample line bundles.** When $\mathcal{L}_{\vec{c}}$ is ample, we may choose h such that ω_h is a symplectic form. Then $\Phi_{\vec{c},h} : X \to M_{\mathbb{R}}$ is the moment map of the $T_{\mathbb{R}}$-action on the symplectic manifold (X, ω_h). The image of the moment map $\Phi_{\vec{c},h}$ is the convex polytope $\triangle_{\vec{c}}$ defined by (5). Note that the moment map $\Phi_{\vec{c},h}$ depends on both $\vec{c}$ and h, but the moment polytope $\triangle_{\vec{c}} = \Phi_{\vec{c},h}(X)$ depends on $\vec{c}$ but not on h. $\Phi_{\vec{c},h}$ restricts to a homeomorphism $X_{\geq 0} \to \triangle_{\vec{c}}$, and $\Phi_h \circ j_0 : N_{\mathbb{R}} \to M_{\mathbb{R}}$ maps $N_{\mathbb{R}}$ diffeomorphically to $\triangle_{\vec{c}}^\circ$, the interior of the moment polytope $\triangle_{\vec{c}} \subset M_{\mathbb{R}}$. Let $\Psi_{\vec{c},h} : N_{\mathbb{R}} \to \triangle_{\vec{c}}^\circ$ be this diffeomorphism. Then $\mathbb{L}_{\vec{c},h}$ can be rewritten as a graph over $\triangle_{\vec{c}}^\circ$:

$$\mathbb{L}_{\vec{c},h} = \{(x, \Psi_{\vec{c},h}^{-1}(x)) \mid x \in \triangle_{\vec{c}}^\circ\} \subset \triangle_{\vec{c}}^\circ \times N_{\mathbb{R}} = T^*\triangle_{\vec{c}}^\circ \subset T^*M_{\mathbb{R}}.$$

There exists a real analytic function $f_{\vec{c},h}^* : \triangle_{\vec{c}}^\circ \to \mathbb{R}$, unique up to addition of a constant $r \in \mathbb{R}$, such that $\Psi_{\vec{c},h}^{-1}(x) = df_{\vec{c},h}^*(x)$. Indeed $f_{\vec{c},h}^* : \triangle_{\vec{c}}^\circ \to \mathbb{R}$ can be chosen to be the *Legendre transform* of $f_h : N_{\mathbb{R}} \to \mathbb{R}$. More explicitly, let $\langle\ ,\ \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ be the natural pairing. Then

$$f_{\vec{c},h}^*(x) = \sup_{y \in N_{\mathbb{R}}} (\langle x, y \rangle - f_{\vec{c},h}(y)), \quad x \in \triangle_{\vec{c}}^\circ. \tag{25}$$

We now consider the equivariant anti-ample line bundle $\mathcal{L}_{\vec{c}}^{-1} = \mathcal{L}_{-\vec{c}}$ equipped with the $T_{\mathbb{R}}$-invariant, real analytic hermitian metric h^{-1}. Then $\Phi_{-\vec{c},h^{-1}} = -\Phi_{\vec{c},h}$, so

$$\triangle_{-\vec{c}} := \Phi_{-\vec{c},h^{-1}}(X) = -\triangle_{\vec{c}} = \{m \in M_{\mathbb{R}} \mid \langle m, v_i\rangle \leq c_i, i = 1, \ldots, r\},$$

and

$$\mathbb{L}_{-\vec{c},h^{-1}} = \{(-\Psi_{\vec{c},h}(y), y) \mid y \in N_{\mathbb{R}}\} = \{(x, \Psi_{\vec{c},h}^{-1}(-x)) \mid x \in \Delta^{\circ}_{-\vec{c}}\}.$$

Define a map $\beta : M_{\mathbb{R}} \times N_{\mathbb{R}} \to M_{\mathbb{R}} \times N_{\mathbb{R}}$ by $\beta(x, y) = (-x, y)$. It is easy to see that $\mathbb{L}_{-\vec{c},h^{-1}} = \beta(\mathbb{L}_{\vec{c},h})$.

5. T-dual Lagrangians as Objects in the Fukaya Category

The goal of this section is to prove Theorem 3. Let X_Σ be a smooth projective toric variety defined by a fan $\Sigma \subset N_{\mathbb{R}}$. Let $\mathcal{L}_{\vec{c}}$ be an equivariant ample line bundle on X_Σ, and let $\mathbb{L}_{\vec{c},h}$ and $\mathbb{L}_{-\vec{c},h^{-1}}$ be as in Section 4.3. In Section 5.1, we prove that $\mathbb{L}_{\vec{c},h}$ and $\mathbb{L}_{-\vec{c},h^{-1}}$ are objects in $Fuk(T^*M_{\mathbb{R}}; \Lambda_\Sigma)$. In Section 5.2, we prove that (see Theorem 5.10)

$$\mathbb{L}_{\vec{c},h} \cong \tau(\mathcal{L}_{\vec{c}}), \quad \mathbb{L}_{-\vec{c},h^{-1}} \cong \tau(\mathcal{L}_{-\vec{c}}). \tag{26}$$

where $\tau = \mu \circ \kappa$ is the composition of the microlocalization μ and the coherent-constructible correspondence κ.

5.1. **T-dual Lagrangians are branes.** In this section, we study the behavior of T-dual Lagrangians on the compactification $D^*M_{\mathbb{R}} = \overline{T}^* M_{\mathbb{R}}$ in the cotangent. We will show that Lagrangians $\mathbb{L}_{-\vec{c},h}$ from anti-ample line bundles $\mathcal{L}_{-\vec{c}}$ are branes (Proposition 5.7); as an immediate consequence, Lagrangians $\mathbb{L}_{\vec{c},h}$ from ample line bundles $\mathcal{L}_{\vec{c}}$ are also branes (Corollary 5.8). To prove a Lagrangian L is a brane of $Fuk(T^*M_{\mathbb{R}}; \Lambda_\Sigma)$, we need to establish that (1) L is tame, (2) L has a brane structure, (3) $\pi(L)$ is bounded, (4) $\overline{L} \subset \overline{T}^* M_{\mathbb{R}}$ is a $\mathcal{C}$-set, (5) $L^\infty \subset \Lambda_\Sigma^\infty$.

Proposition 5.1 (T-dual Lagrangians are tame). *Let $\mathbb{L}_{\vec{c},h}$ be the T-dual Lagrangian constructed in 4.1. (We do not assume $\mathcal{L}_{\vec{c}}$ is ample or anti-ample.) Then:*

1. *there exists $\rho > 0$ such that for every $p \in \mathbb{L}_{\vec{c},h^{-1}}$, the set of points $p' \in \mathbb{L}_{\vec{c},h^{-1}}$ with $d(p, p') < \rho$ is contractible;*
2. *there exists a constant $C = C(\vec{c}, h)$ such that*
$$d_{\mathbb{L}_{\vec{c},h^{-1}}}(p, p') < Cd(p, p')$$
for all $p, p' \in \mathbb{L}_{\vec{c},h^{-1}}$, where d is the distance in $T^*M_{\mathbb{R}}$ and $d_{\mathbb{L}_{\vec{c},h}}$ is the distance in $\mathbb{L}_{\vec{c},h}$.

Therefore $\mathbb{L}_{\vec{c},h}$ is tame in the sense of [NZ].

Proof. The Lagrangian $\mathbb{L}_{\vec{c},h}$ is the graph of the map $\Phi_{\vec{c},h} \circ j_0 : N_{\mathbb{R}} \to M_{\mathbb{R}}$. We first show that the first and second derivatives of $\Phi_{\vec{c},h} \circ j_0$, i.e. $\frac{\partial^2 f_{\vec{c},h}}{\partial y_i \partial y_j}$ and $\frac{\partial^3 f_{\vec{c},h}}{\partial y_i \partial y_j \partial y_l}$ are bounded for any i, j, l.

For each top dimensional cone $C_k \in \Sigma$, $k = 1, \ldots, v$, the associated affine toric variety $U_k \cong \mathbb{C}^n$ is smooth since X_Σ is a smooth projective toric variety. The coordinates in U_k are given by

$$z_{k,i} = s_{k,i} + \sqrt{-1}t_{k,i} = r_{k,i}\exp(\sqrt{-1}\theta_{k,i}) = \exp(y_{k,i} + \sqrt{-1}\theta_{k,i}).$$

Notice that the coordinates $y_{k,i}$ and y_i differ by a linear change of basis. Fix a compact part $U_k' = \{|z_{k,1}|^2 + \cdots + |z_{k,n}|^2 \leq M\} \subset U_k$ such that $X_\Sigma = \cup_{k=1}^v U_k'$. The 2-form

$$\begin{aligned}
\omega_h &= \sum_{i,j=1}^n \frac{\partial^2 f_{\vec{c},h}}{\partial y_{k,i}\partial y_{k,j}} dy_{k,i} \wedge d\theta_{k,j} = \sum_{i,j=1}^n \frac{\partial^2 f_{\vec{c},h}}{r_{k,i}r_{k,j}\partial y_{k,i}\partial y_{k,j}} r_{k,j} dr_{k,i} \wedge d\theta_{k,j} \\
&= \sum_{i,j=1}^n \cos(\theta_{k,i} - \theta_{k,j}) \cdot \frac{\partial^2 f_{\vec{c},h}}{r_{k,i}r_{k,j}\partial y_{k,i}\partial y_{k,j}} \cdot (ds_{k,i} \wedge dt_{k,j} + ds_{k,j} \wedge dt_{k,i}) \\
&+ \sum_{i,j=1}^n \sin(\theta_{k,i} - \theta_{k,j}) \cdot \frac{\partial^2 f_{\vec{c},h}}{r_{k,i}r_{k,j}\partial y_{k,i}\partial y_{k,j}} \cdot (ds_{k,i} \wedge ds_{k,j} + dt_{k,i} \wedge dt_{k,j}).
\end{aligned}$$

Hence ω_h must be in the form $\omega_h = a_{k,ij}(ds_{k,i} \wedge dt_{k,j} + ds_{k,j} \wedge dt_{k,i}) + b_{k,ij}(ds_{k,i} \wedge ds_{k,j} + dt_{k,i} \wedge dt_{k,j})$, and we know that $a_{k,ij}$ and $b_{k,ij}$ are bounded in U_k' since they are real analytic functions on U_k. By comparing with the expression above,

$$\frac{\partial^2 f_{\vec{c},h}}{\partial y_{k,i}\partial y_{k,j}} = \frac{a_{k,ij} r_{k,i} r_{k,j}}{\cos(\theta_{k,i} - \theta_{k,j})} = \frac{b_{k,ij} r_{k,i} r_{k,j}}{\sin(\theta_{k,i} - \theta_{k,j})}.$$

Thus

$$\left| \frac{\partial^2 f_{\vec{c},h}}{\partial y_{k,i}\partial y_{k,j}} \right| \leq \sqrt{2} \max\{|a_{k,ij}|, |b_{k,ij}|\} \cdot r_{k,i} r_{k,j}.$$

The right hand side is bounded on U_k', and therefore $\frac{\partial^2 f_{\vec{c},h}}{\partial y_{k,i}\partial y_{k,j}}$ is bounded on U_k', for any i, j.

Moreover,

$$\frac{\partial^3 f_{\vec{c},h}}{\partial y_{k,i}\partial y_{k,j}\partial y_{k,l}} = \frac{1}{\cos(\theta_{k,i} - \theta_{k,j})} \frac{\partial(a_{k,ij} r_{k,i} r_{k,j})}{\partial y_{k,l}} = \frac{1}{\sin(\theta_{k,i} - \theta_{k,j})} \frac{\partial(b_{k,ij} r_{k,i} r_{k,j})}{\partial y_{k,l}}$$

also implies that on U_k' the derivatives $\frac{\partial^3 f_{\vec{c},h}}{\partial y_{k,i}\partial y_{k,j}\partial y_{k,l}}$ are bounded since $\frac{\partial(a_{k,ij} r_{k,i} r_{k,j})}{\partial y_{k,l}}$ and $\frac{\partial(b_{k,ij} r_{k,i} r_{k,j})}{\partial y_{k,l}}$ are bounded on U_k'.

There exists constants (C_{ij}^k), $k = 1, \ldots, v$, such that

$$\frac{\partial^2 f_{\vec{c},h}}{\partial y_i \partial y_j} = \sum_{a,b} C_{ia}^k C_{jb}^k \frac{\partial^2 f_{\vec{c},h}}{\partial y_{k,a}\partial y_{k,b}}; \quad \frac{\partial^3 f_{\vec{c},h}}{\partial y_i \partial y_j \partial y_l} = \sum_{a,b,c} C_{ia}^k C_{jb}^k C_{lc}^k \frac{\partial^3 f_{\vec{c},h}}{\partial y_{k,a}\partial y_{k,b}\partial y_{k,c}}.$$

Hence there is M_k such that $\left|\frac{\partial^2 f_h}{\partial y_i \partial y_j}\right| < M_k$ and $\left|\frac{\partial^3 f_h}{\partial y_i \partial y_j \partial y_l}\right| < N_k$ on U_k' for any i, j. By construction $\cup_{k=1}^v U_k' = X_\Sigma$. It follows that for $M = \max M_k$ and $N = \max N_k$, we have the inequalities

$$\left|\frac{\partial^2 f_{\vec{c},h}}{\partial y_i \partial y_j}\right| < M; \quad \left|\frac{\partial^3 f_{\vec{c},h}}{\partial y_i \partial y_j \partial y_l}\right| < N,$$

for any i, j.

To show (1), let $p = (x_0, y_0)$ be any point in $\mathbb{L}_{\vec{c},h}$. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a unit vector in $N_\mathbb{R}$, and $y_t = y_0 + t\xi$. Set $p_t = (x_t, y_t) \in \mathbb{L}_{\vec{c},h}$ where $x_t = \Phi_{\vec{c},h} \circ j_0(y_t)$. Near p the Taylor theorem gives

$$x_t = x_0 + tA + t^2 B(t'),$$

where A, B are in $M_{\mathbb{R}}$ with each component

$$A_i = \sum_{j=1}^{n} \xi_j \frac{\partial^2 f}{\partial y_i \partial y_j}(y_0),\ \ B_i = \sum_{j,l=1}^{n} \xi_j \xi_l \frac{\partial^3 f}{\partial y_i \partial y_j \partial y_l}(x_{t'}),$$

and $t' \in [0,t]$ depends on t. Therefore, $d(p,p_t)^2 = t^2 + (tA + t^2B(t'))^2$. Since by our estimates $|A| < nM$ and $|B| < n^2N^2$, there exists an $\rho > 0$ such that for any direction ξ, $d(p,p_t)$ increases as long as $0 < t < \rho$. Hence the set $\{p' \in \mathbb{L}_{\vec{c},h} : d(p,p') < \rho\} \subset \{p' \in \mathbb{L}_{\vec{c},h} : d_{N_{\mathbb{R}}}(p,p') < \rho\}$ is a star-set, and it is contractible.

For any $p_1 = (x_1,y_1), p_2 = (x_2,y_2) \in \mathbb{L}_{\vec{c},h}$,

$$d_{\mathbb{L}_{\vec{c},h}}(p_1,p_2) \leq \int_{l_{y_1,y_2}} \sqrt{1+nM^2}d\xi = \sqrt{1+nM^2}d_{N_{\mathbb{R}}}(y_1,y_2) \leq \sqrt{1+nM^2}d(p_1,p_2),$$

where $d\xi$ is the standard measure on the segment l_{y_1,y_2} from y_1 to y_2 in $N_{\mathbb{R}}$. This shows (2). $\square$

Remark 5.2. In [NZ], a new metric g_{con}, which is the metric of a cone over the spherical bundle $S^*M_{\mathbb{R}}$ near the infinity, is introduced in order to ensure a tame perturbation for any standard Lagrangian. It is no longer needed here since our T-dual Lagrangians are already tame in the usual Sasaki metric. Moreover, we only consider standard or costandard Lagrangians over convex polytopes, which are also tame in the Sasaki metric. Any convex polytope is prescribed by a collection of linear functions $f_i \geq 0$ for $i = 1, \ldots, k$. The standard Lagrangian over it can be written as the graph of $d\log m_1 + \cdots + d\log m_k$, where m_i is a piecewise linear function on $M_{\mathbb{R}}$ which is f_i on the half plane $\{f_i \geq 0\}$ and zero otherwise. The tameness of this standard Lagrangian follows from the tameness of each $d\log m_i$.

From now on, we assume that $\mathcal{L}_{\vec{c}}$ is an equivariant ample line bundle and ω_h is symplectic.

Lemma 5.3 (Compact horizontal support and brane structure)**.** $\mathbb{L}_{-\vec{c},h^{-1}}$ *and* $\mathbb{L}_{\vec{c},h}$ *are horizontally compact Lagrangians inside* $T^*M_{\mathbb{R}}$*, and have canonical brane structures.*

Proof. Horizontal compactness is immediate, as $\triangle_{\vec{c}}$ and $\triangle_{-\vec{c}}$ are bounded. Recall that a brane structure is a relative pin structure and a choice of grading (see [S2] as quoted in [NZ]). Since $\mathbb{L}_{-\vec{c},h^{-1}}$ is the graph of a differential $df^*_{-\vec{c},h^{-1}}$, for $f^*_{-\vec{c},h^{-1}} : \triangle^\circ_{-\vec{c}} \to \mathbb{R}$ (see Section 4.3), it is Hamiltonian isotopic to the zero section.[7] Since $\triangle^\circ_{-\vec{c}} \subset 0_{T^*M_{\mathbb{R}}}$ is a contractible subset of the zero section, it has trivial pin structure and can be given the zero grading. The same goes for $\mathbb{L}_{\vec{c},h}$. $\square$

We use the notation of Section 2.4. Given a cone $\tau \in \Sigma$, define $\mathcal{U}_{\tau,\pm\vec{c}} = \pm\Phi_{\vec{c},h}(O^+_\tau) = \pm\Phi_{\vec{c},h}(X) \subset \triangle_{\pm\vec{c}}$, where O^+_τ is defined in Section 2.4, and define $F_{\tau,\pm}$ to be the closures of $\mathcal{U}_{\tau,\pm\vec{c}}$ in $M_{\mathbb{R}}$. Then

$$\begin{aligned}
\mathcal{U}_{\tau,\vec{c}} &= \{m \in \triangle_{\vec{c}} \mid \langle m, v_i\rangle = -c_i \Leftrightarrow v_i \in \tau\} \\
&= \{m \in M_{\mathbb{R}} \mid \langle m, v_i\rangle = -c_i \text{ (resp. } > -c_i) \text{ if } v_i \in \tau \text{ (resp. } \notin \tau)\} \\
F_{\tau,\vec{c}} &= \{m \in \triangle_{\vec{c}} \mid \langle m, v_i\rangle = -c_i \text{ if } v_i \in \tau\} \\
&= \{m \in M_{\mathbb{R}} \mid \langle m, v_i\rangle = -c_i \text{ (resp. } \geq -c_i) \text{ if } v_i \in \tau \text{ (resp. } \notin \tau)\}
\end{aligned}$$

[7] The isotopy is achieved by he Hamiltonian flow of the function $H = f^*_{-\vec{c},h^{-1}} \circ \pi$, which takes $\mathbb{L}_{-\vec{c},h^{-1}}$ to $(1-t)\mathbb{L}_{-\vec{c},h^{-1}}$ in time t. The subset $\triangle^\circ_{-\vec{c}}$ of the zero section is the image of time-one flow.

In particular, $\mathcal{U}_{\tau,\pm\vec{c}}$ are contractible open subsets of an affine subspace of $M_{\mathbb{R}}$, and

$$\mathcal{U}_{\{0\},\pm\vec{c}} = \triangle^\circ_{\pm\vec{c}}, \quad F_{\{0\},\pm\vec{c}} = \triangle_{\pm\vec{c}}.$$

We have a stratification

$$\triangle_{\pm\vec{c}} = \bigcup_{\tau\in\Sigma} \mathcal{U}_{\tau,\pm\vec{c}}.$$

Given a d-dimensional cone $\tau \in \Sigma$, $F_{\tau,\pm\vec{c}}$ is an $(n-d)$-dimensional face of the convex polytope $\triangle_{\pm\vec{c}} \subset M_{\mathbb{R}}$, and has the further stratification

$$F_{\tau,\pm\vec{c}} = \bigcup_{\tau\subset\sigma} \mathcal{U}_{\sigma,\pm\vec{c}}.$$

Let N_τ be the rank d sublattice of N generated by $\tau\cap N$, and let $(N_\tau)_{\mathbb{R}} = N_\tau\otimes\mathbb{R} \cong \mathbb{R}^d$. Let $w_1,\dots,w_d$ be defined as in Section 2.4, so that

$$\tau = \Big\{\sum_{j=1}^d r_j w_j \mid r_j \geq 0\Big\}, \quad (N_\tau)_{\mathbb{R}} = \Big\{\sum_{j=1}^d r_j w_j \mid r_j \in \mathbb{R}\Big\}.$$

The conormal bundle of $\mathcal{U}_{\tau,\pm\vec{c}} \subset M_{\mathbb{R}}$ is

$$T^*_{\mathcal{U}_{\tau,\pm\vec{c}}} M_{\mathbb{R}} = \mathcal{U}_{\tau,\pm\vec{c}} \times (N_\tau)_{\mathbb{R}} \subset M_{\mathbb{R}} \times N_{\mathbb{R}} = T^*M_{\mathbb{R}}.$$

Its closure is the conormal bundle of $F_{\tau,\pm\vec{c}}$:

$$T^*_{F_{\tau,\pm\vec{c}}} M_{\mathbb{R}} = F_{\tau,\pm\vec{c}} \times (N_\tau)_{\mathbb{R}}.$$

Let $\Sigma' = \cup_{d>0}\Sigma(d)$, so that $\Sigma = \{\{0\}\}\cup\Sigma'$. Define a conical Lagrangian $\Lambda_{\pm\vec{c}} \subset T^*M_{\mathbb{R}}$ by

$$\Lambda_{\pm\vec{c}} := \mathcal{U}_{\{0\},\pm\vec{c}} \times \{0\} \cup \bigcup_{\tau\in\Sigma'} \mathcal{U}_{\tau,\pm\vec{c}} \times (-\tau^\circ) = \bigcup_{\tau\in\Sigma} F_{\tau,\pm\vec{c}} \times (-\tau)$$

Each $F_{\tau,\pm\vec{c}} \times (-\tau)$ is a closed subanalytic subset of $T^*M_{\mathbb{R}}$. Note that

$$\Lambda_{\pm\vec{c}} \subset \Lambda_\Sigma.$$

Let $\iota : T^*M_{\mathbb{R}} \to D^*M_{\mathbb{R}}$ be defined as in (12). Define

$$\mathbb{L}^\infty_{\pm\vec{c},h^{\pm1}} := \overline{\iota(\mathbb{L}_{\pm\vec{c},h^{\pm1}})} \cap T^\infty M_{\mathbb{R}}, \quad \Lambda^\infty_{\pm\vec{c}} := \overline{\iota(\Lambda_{\pm\vec{c}})} \cap T^\infty M_{\mathbb{R}}.$$

Then

$$\Lambda^\infty_{\pm\vec{c}} = \bigcup_{\tau\in\Sigma'} \mathcal{U}_{\tau,\pm\vec{c}} \times ((-\tau^\circ)\cap S(N_{\mathbb{R}})) = \bigcup_{\tau\in\Sigma'} F_{\tau,\pm\vec{c}} \times ((-\tau)\cap S(N_{\mathbb{R}}))$$

where $S(N_{\mathbb{R}}) = \{y \in N_{\mathbb{R}} \mid |y|_{N_{\mathbb{R}}} = 1\} \cong S^{n-1}$.

We now introduce an analytic-geometric category. (See Section A.1 for a brief review of analytic-geometric categories.)

Definition 5.4. Define $f : \mathbb{R} \to (-1,1)$ by

$$f(t) = \begin{cases} e^{-1/t} & t > 0, \\ 0 & t = 0, \\ -e^{1/t} & t < 0. \end{cases} \tag{27}$$

Let $\mathcal{C}$ be the smallest analytic-geometric category such that f is a $\mathcal{C}$-map.

Remark 5.5. Let f be defined by (27). Then f is C^∞ on $\mathbb{R}$, is real analytic on $\mathbb{R}\setminus\{0\}$, and is a homeomorphism from $\mathbb{R}$ to $(-1,1)$. So $f^{-1} : (-1,1) \to \mathbb{R}$ is a $\mathcal{C}$-map, and f is an $\mathcal{C}$-isomorphism.

Proposition 5.6. *$\mathbb{L}^{\infty}_{-\vec{c},h^{-1}} = \Lambda^{\infty}_{-\vec{c}}$, and $\overline{\iota(\mathbb{L}_{-\vec{c},h^{-1}})} \subset D^*M_{\mathbb{R}} = \overline{T}^* M_{\mathbb{R}}$ is a $\mathcal{C}$-set, where $\mathcal{C}$ is the analytic-geometric category defined in Definition 5.4.*

Proof. The proof is given in Section A.2. □

Corollary 5.7 (T-dual Lagrangians are branes)**.** *T-dual Lagrangians from anti-ample equivariant line bundles are branes. That is, $\mathbb{L}_{-\vec{c},h^{-1}}$ defines an object of $Fuk(T^*M_{\mathbb{R}};\Lambda_{\Sigma})$.*

Proof. First, we put the trivial vector bundle on $\mathbb{L}_{-\vec{c},h^{-1}}$. The existence of tame perturbations follows from Proposition 5.1 since one may choose the constant "perturbation". The remaining conditions on branes are assured by Lemma 5.3 and 5.6. □

Since the involution $\beta : M_{\mathbb{R}} \times N_{\mathbb{R}} \to M_{\mathbb{R}} \times N_{\mathbb{R}}$ given by $(x, y) \mapsto (-x, y)$ is a $\mathcal{C}$-isomorphism such that $\beta(\Lambda_{\Sigma}) = \Lambda_{\Sigma}$, and the tameness is obviously preserved, we have the immediate corollary:

Corollary 5.8. *T-dual Lagrangians from ample line bundles are branes: $\mathbb{L}_{\vec{c},h}$ defines an object of $Fuk(T^*M_{\mathbb{R}};\Lambda_{\Sigma})$.*

5.2. **T-dual Lagrangians of ample bundles are costandard branes.** Having shown $\mathbb{L}_{-\vec{c},h^{-1}}$ is a brane, we now relate it to the *standard* brane associated to $\triangle^{\circ}_{-\vec{c}}$. The key is to study normalized geodesic flow at infinity, which controls the hom spaces of Lagrangians which intersect at infinity. The symplectomorphism of inversion on the fibers intertwines with Verdier duality of constructible sheaves under microlocalization [N1]. We use this fact to relate $\mathbb{L}_{\vec{c},h}$ to the *costandard* brane on the set $\triangle^{\circ}_{\vec{c}}$.

5.2.1. *Normalized geodesic flow.* Let $\{e^*_i\}$, $\{e_j\}$ be dual orthonormal bases on M and N, respectively (as in Sec. 3.2), and let x_i, y_j be associated real coordinates. We can equate e_j with dx^j, so $(x, y) = (\sum_i x_i e^*_i, \sum_j y_j dx_j) \in M_{\mathbb{R}} \times N_{\mathbb{R}} = T^*M_{\mathbb{R}}$. The inner product on $N_{\mathbb{R}}$ induces a linear isomorphism $I : N_{\mathbb{R}} \to M_{\mathbb{R}}$ given by $y \mapsto \sum_{j=1}^{n} \langle e^*_j, y\rangle e^*_j$. In particular, $I(e_i) = e^*_i$, so I is an isometry. Define $y^* = I(y)$.

Given a vector space V, let $V' = V \setminus \{0\}$; given a vector bundle E, let E' denote the complement of the zero section. The normalized geodesic flow on $(T^*M_{\mathbb{R}})' \cong (TM_{\mathbb{R}})'$ is given by

$$\gamma_t : (T^*M_{\mathbb{R}})' \cong M_{\mathbb{R}} \times N'_{\mathbb{R}} \to (T^*M_{\mathbb{R}})' \cong M_{\mathbb{R}} \times N'_{\mathbb{R}},$$
$$\gamma_t(x, y) = (x + \frac{ty^*}{|y^*|_{M_{\mathbb{R}}}}, y),$$

where $|y^*|_{M_{\mathbb{R}}} = |y|_{N_{\mathbb{R}}}$ because $y \mapsto y^*$ is an isometry from $N_{\mathbb{R}}$ to $M_{\mathbb{R}}$.

Let $\mathcal{L}_{\vec{c}}$ be an ample line bundle, and let h, Φ_h, $\triangle_{-\vec{c}}$, $\mathbb{L}_{-\vec{c},h^{-1}}$, etc. be defined as in Section 4.3. Let $q \in \partial\triangle_{-\vec{c}} \subset M_{\mathbb{R}}$ be a boundary point of the polytope. We consider the following two Lagrangians in $T^*M_{\mathbb{R}} \cong M_{\mathbb{R}} \times N_{\mathbb{R}}$:

$$\begin{aligned} L_q &= \{(q, y) \mid y \in N_{\mathbb{R}}\} \subset M_{\mathbb{R}} \times N_{\mathbb{R}} \\ \mathbb{L}_{-\vec{c},h^{-1}} &= \{(-\Phi_h \circ j_0(y), y) \mid y \in N_{\mathbb{R}}\} \subset M_{\mathbb{R}} \times N_{\mathbb{R}} \end{aligned}$$

Let $L'_q = (T^*M_{\mathbb{R}})' \cap L_q$, and let $\mathbb{L}'_{-\vec{c},h^{-1}} = (T^*M_{\mathbb{R}})' \cap \mathbb{L}_{-\vec{c},h^{-1}}$. Then

$$\begin{aligned} \gamma_t(L'_q) &= \{(q + \frac{ty^*}{|y^*|_{M_{\mathbb{R}}}}, y) \mid y \in N'_{\mathbb{R}}\} \subset M_{\mathbb{R}} \times N'_{\mathbb{R}} \\ \gamma_t(\mathbb{L}'_{-\vec{c},h^{-1}}) &= \{(-\Phi_h \circ j_0(y) + \frac{ty^*}{|y^*|_{M_{\mathbb{R}}}}, y) \mid y \in N'_{\mathbb{R}}\} \subset M_{\mathbb{R}} \times N'_{\mathbb{R}}. \end{aligned}$$

Note that $(x, y) \in \gamma_{t_1}(L_q) \cap \gamma_{t_2}(\mathbb{L}_{-\vec{c},h^{-1}})$ if and only if

$$y \in N'_{\mathbb{R}}, \quad q + \Phi_h \circ j_0(y) = \frac{(t_2 - t_1)y^*}{|y^*|_{M_{\mathbb{R}}}}. \tag{28}$$

Lemma 5.9. *Given any $q \in \partial\triangle_{-\vec{c}}$, there exists $\delta > 0$ such that*

$$0 \le t_1 \le t_2 < \delta \Rightarrow \gamma_{t_1}(L'_q) \cap \gamma_{t_2}(\mathbb{L}'_{-\vec{c},h^{-1}}) = \emptyset.$$

Proof. We use the notation in Section 5.1.

$$\triangle_{-\vec{c}} = \bigcup_{\tau \in \Sigma} \mathcal{U}_{\tau,-\vec{c}}.$$

The right hand side is a disjoint union. Let $\Sigma(d)$ be the set of d-dimensional cones in Σ.

Step 1. $q \in \partial\triangle_{-\vec{c}}$, so there exists a unique $d > 0$ and a unique $\tau \in \Sigma(d)$ such that $q \in \mathcal{U}_{\tau,-\vec{c}} = -\Phi_h(\mathcal{O}^+_\tau)$. There exists a unique $x \in \mathcal{O}^+_\tau \subset (X_\Sigma)_{\ge 0}$ such that $-\Phi_h(x) = q$.

There exists $\sigma \in \Sigma(n)$ such that $\tau \subset \sigma$. Let w_j be defined as in the proof of Proposition 5.6 (see Section A.2), so

$$\tau = \{r_1 w_1 + \cdots + r_d w_d \mid r_j \ge 0\}, \quad \sigma = \{r_1 w_1 + \cdots + r_n w_n \mid r_j \ge 0\}$$

The holomorphic coordinates of $X_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}^n$ are $Z_j = \chi^{w_j^\vee}$, $j = 1, \ldots, n$. There exist $b_{d+1}, \ldots, b_n \in \mathbb{R}$ such that the coordinates of $x \in U_\sigma$ are given by

$$Z_1 = \cdots = Z_d = 0, \quad Z_{d+1} = e^{b_{d+1}}, \ldots, Z_n = e^{b_n}.$$

Step 2. For any $r > 0$, define

$$S_r = \{r_1 w_1 + \cdots + r_n w_n \mid r_i \in (-r, r)\}, \quad B_r = \{y \in N_{\mathbb{R}} \mid |y|_{N_{\mathbb{R}}} < r\}.$$

There exists $c \in (0, 1)$ such that for all $r > 0$,

$$B_{cr} \subset S_r \subset B_{c^{-1}r}.$$

Let $R = \max\{|b_{d+1}|, \ldots, |b_n|\} + 1$. Note that x is contained in (see Section 2.4 for definitions)

$$X^+_\tau := X_\tau \cap (X_\Sigma)_{\ge 0} \cong [0, \infty)^d \times (\mathbb{R}^+)^{n-d}$$

which is an open set in $(X_\Sigma)_{\ge 0}$. A neighborhood of x in X^+_τ is given by

$$U = \{(Z_1, \ldots, Z_n) \mid Z_1, \ldots, Z_d \in [0, e^{-2c^{-2}R}), \quad Z_{d+1}, \ldots, Z_n \in (e^{-R}, e^R)\}.$$

Recall that $j_0 : N_{\mathbb{R}} \to U_\sigma$ is given by $\sum_{j=1}^n r_j w_j \mapsto (e^{r_1}, \ldots, e^{r_n})$, so

$$\begin{aligned} j_0^{-1}(U) &= \{r_1 w_1 + \cdots + r_n w_n \mid r_1, \ldots, r_d < -2c^{-2}R, \quad r_{d+1}, \ldots, r_n \in (-R, R)\} \\ &\cong (-\infty, -2c^{-2}R)^d \times (-R, R)^{n-d}. \end{aligned}$$

Step 3. $-\Phi_h$ maps X^+_τ homeomorphically to $-\Phi_h(X^+_\tau)$, so there exists $\delta > 0$ such that $B(q, \delta) := \{m \in M_{\mathbb{R}} \mid |m - q|_{M_{\mathbb{R}}} < \delta\} \subset -\Phi_h(U)$.

Claim: For any $y \in N'_{\mathbb{R}}$ and $0 \le t_1 \le t_2 < \delta$, (28) does not hold. Therefore,

$$\gamma_{t_1}(L'_q) \cap \gamma_{t_2}(\mathbb{L}'_{-\vec{c},h^{-1}}) = \emptyset.$$

Case 1. $j_0(y) \notin U$. Then $-\Phi_h \circ j_0(y) \notin B(q,\delta)$, so

$$|q + \Phi_h \circ j_0(y)|_{M_{\mathbb{R}}} = |-\Phi_h \circ j_0(y) - q|_{M_{\mathbb{R}}} \ge \delta.$$

On the other hand

$$\left| \frac{(t_2 - t_1)y^*}{|y^*|_{M_{\mathbb{R}}}} \right| = t_2 - t_1 < \delta.$$

So

$$q + \Phi_h \circ j_0(y) \ne \frac{(t_2 - t_1)y^*}{|y^*|_{M_{\mathbb{R}}}}.$$

Case 2. $j_0(y) \in U$. We have

$$y = r_1 w_1 + \cdots + r_n w_n, \quad r_1, \ldots, r_d < -2c^{-2}R, \quad r_{d+1}, \ldots, r_n \in (-R, R).$$

Let $y_1 = r_1 w_1 + \cdots + r_d w_d$ and $y_2 = r_{d+1} w_{d+1} + \cdots r_n w_n$. Then

$$y = y_1 + y_2, \quad y_1 \in N_{\mathbb{R}} \setminus \overline{S_{2c^{-2}R}} \subset N_{\mathbb{R}} \setminus \overline{B_{2c^{-1}R}}, \quad y_2 \in S_R \subset B_{c^{-1}R}.$$

Therefore,

$$|y_1|_{N_{\mathbb{R}}} > 2c^{-1}R > c^{-1}R > |y_2|_{N_{\mathbb{R}}}.$$

Let $(v_1, v_2)_{N_{\mathbb{R}}}$ denote the inner product on $N_{\mathbb{R}}$, so that

$$(e_i, e_j)_{N_{\mathbb{R}}} = \delta_{ij}, \quad \langle v_1^*, v_2 \rangle = (v_1, v_2)_{N_{\mathbb{R}}}.$$

Then

$$\langle \frac{(t_2 - t_1)y^*}{|y^*|_{M_{\mathbb{R}}}}, y_1 \rangle = \frac{t_2 - t_1}{|y^*|_{M_{\mathbb{R}}}} (y_1 + y_2, y_1)_{N_{\mathbb{R}}}$$

where $t_2 - t_1 \ge 0$, and

$$\begin{aligned} (y_1 + y_2, y_1)_{N_{\mathbb{R}}} &= |y_1|^2_{N_{\mathbb{R}}} + (y_2, y_1)_{N_{\mathbb{R}}} \ge |y_1|^2_{N_{\mathbb{R}}} - |y_2|_{N_{\mathbb{R}}} |y_1|_{N_{\mathbb{R}}} \\ &= |y_1|_{N_{\mathbb{R}}} (|y_1|_{N_{\mathbb{R}}} - |y_2|_{N_{\mathbb{R}}}) > 0. \end{aligned}$$

So

$$\langle \frac{(t_2 - t_1)y^*}{|y^*|_{M_{\mathbb{R}}}}, y_1 \rangle \ge 0. \tag{29}$$

On the other hand,

$$\langle q + \Phi_h \circ j_0(y), y_1 \rangle = \sum_{j=1}^{d} r_j \langle q + \Phi_h \circ j_0(y), w_j \rangle. \tag{30}$$

Let $w_j = v_{i(j)}$. Since $q \in \mathcal{U}_{\tau,-\vec{c}}$ and $-\Phi_h \circ j_0(y) \in \triangle^\circ_{-\vec{c}}$, for $j = 1, \ldots, d$,

$$\langle q, w_j \rangle = c_{i(j)}, \quad \langle -\Phi_h \circ j_0(y), w_j \rangle < c_{i(j)}.$$

So we have

$$\langle q + \Phi_h \circ j_0(y), w_j \rangle > 0, \quad r_j < -2c^{-1}R < 0, \tag{31}$$

Equations (30) and (31) imply

$$\langle q + \Phi_h \circ j_0(y), y_1 \rangle < 0. \tag{32}$$

Combining (29) and (32), we see that

$$q + \Phi_h \circ j_0(y) \ne \frac{(t_2 - t_1)y^*}{|y^*|_{M_{\mathbb{R}}}}.$$

□

5.2.2. *Lagrangians from anti-ample line bundles are standard branes.* We now show $\mathbb{L}_{-\vec{c},h^{-1}}$ is isomorphic to the standard Lagrangian brane over $\triangle^\circ_{-\vec{c}}$, and that $\mathbb{L}_{\vec{c},h}$ is isomorphic to the costandard brane over $\triangle^\circ_{\vec{c}}$.

Theorem 5.10. *Let $\mu : Sh_{cc}(M_\mathbb{R}; \Lambda_\Sigma) \to Fuk(T^*M_\mathbb{R}; \Lambda_\Sigma)$ be the microlocalization quasi-embedding of Theorem 3.4. Then $\mathbb{L}_{-\vec{c},h^{-1}} \cong \mu(i_*\mathbb{C}_{\triangle^\circ_{-\vec{c}}})$, and $\mathbb{L}_{\vec{c},h} \cong \mu(i_!\omega_{\triangle^\circ_{\vec{c}}})$.*

Proof. We show $\mathbb{L}_{-\vec{c},h^{-1}} \cong \mu(i_*\mathbb{C}_{\triangle^\circ_{-\vec{c}}})$ by proving, following [N1], that the two objects define isomorphic modules under the Yoneda embedding

$$\mathcal{Y} : DFuk(T^*M_\mathbb{R}) \to mod(DFuk(T^*M_\mathbb{R})), \qquad \mathcal{Y}(L) = hom_{DFuk(T^*M_\mathbb{R})}(-, L).$$

To prove that $\mathcal{Y}(\mathbb{L}_{-\vec{c},h^{-1}}) \cong \mathcal{Y}(\mu(i_*\mathbb{C}_{\triangle^\circ_{-\vec{c}}}))$, we first fix a triangulation $\mathcal{T}$ of $M_\mathbb{R}$ containing $\{\mathcal{U}_{\tau,-\vec{c}} \mid \tau \in \Sigma\}$ (recall $\mathcal{U}_{\{0\},-\vec{c}} = \triangle^\circ_{-\vec{c}}$). The technique of [N1] exploits the triangulation to resolve the diagonal standard, i.e. the identity functor. What emerges is that the Yoneda module of any object $\mathcal{Y}(L)$ is expressed in terms of (sums and cones of shifts of) Yoneda modules from standards, $\mathcal{Y}(\mu(i_*\mathbb{C}_T))$, where $T \in \mathcal{T}$. The coefficient of the Yoneda standard module $\mathcal{Y}(\mu(i_*\mathbb{C}_T))$, takes the form $hom_{DFuk(T^*M_\mathbb{R})}(L_{\{t\}*}, L)$, where t is any point in T (contractibility of T means that the choice is irrelevant up to isomorphism)—see Remark 4.5.1 of [N1].

We now apply this to $L = \mathbb{L}_{-\vec{c},h^{-1}}$. First consider the case $T \neq \triangle^\circ_{-\vec{c}}$, and let $t \in T$. Then if $T \cap \triangle_{-\vec{c}} = \emptyset$, clearly $hom_{DFuk(T^*M_\mathbb{R})}(L_{\{t\}*}, \mathbb{L}_{-\vec{c},h^{-1}}) = 0$, since $L_{\{t\}*}$ is just the fiber $T^*_t M_\mathbb{R}$. Otherwise, if $T \cap \partial\triangle$ is nonempty, then Proposition 5.9 ensures us that $hom_{DFuk(T^*M_\mathbb{R})}(L_{\{t\}*}, \mathbb{L}_{-\vec{c},h^{-1}}) = 0$. Finally, if $T = \triangle^\circ_{-\vec{c}}$, then since $\mathbb{L}_{-\vec{c},h^{-1}}$ is a graph over T, we have $hom_{DFuk(T^*M_\mathbb{R})}(L_{\{t\}*}, \mathbb{L}_{-\vec{c},h^{-1}}) = \mathbb{C}$. Therefore, $\mathbb{L}_{-\vec{c},h^{-1}} \cong \mu(i_*\mathbb{C}_{\triangle^\circ_{-\vec{c}}})$, and the first statement is proved. Note that the result is independent of how $\mathcal{T}$ was chosen.

The map $\alpha : (x, y) \mapsto (x, -y)$ gives rise to a duality functor (still denoted by α)

$$\alpha : Fuk(T^*M_\mathbb{R})^\circ \to Fuk(T^*M_\mathbb{R}).$$

The functor α sends a Lagrangian brane L to $\alpha(L)$. It is proved in Section 5.1 of [N1] (Proposition 5.1.1) that there is a functor quasi-isomorphism

$$\mu \circ \mathcal{D} \cong \alpha \circ \mu : Sh_{cc}(M_\mathbb{R}) \to TrFuk(T^*M_\mathbb{R}).$$

Define another functor $\nu : Fuk(T^*M_\mathbb{R}) \to Fuk(T^*M_\mathbb{R})$ given by the map

$$M_\mathbb{R} \times N_\mathbb{R} \to M_\mathbb{R} \times N_\mathbb{R}, \quad (x, y) \mapsto (-x, -y).$$

The functor ν maps any standard brane $L(U)$ over the submanifold $U \hookrightarrow M_\mathbb{R}$ to the standard brane $L(-U)$ over $-U$. Let $\mathcal{R}$ be the induced push-forward on $Sh_c(M_\mathbb{R})$ given by the map $x \mapsto -x$. It is obvious that there is an isomorphism of functors:

$$\mu \circ \mathcal{R} \cong \nu \circ \mu : Sh_c(M_\mathbb{R}) \to TrFuk(T^*M_\mathbb{R}).$$

Therefore, the quasi-isomorphism $\mathbb{L}_{-\vec{c},h^{-1}} \cong \mu(i_*\mathbb{C}_{\triangle^\circ_{-\vec{c}}})$ gives rise to

$$\nu(\mathbb{L}_{-\vec{c},h^{-1}}) \cong \nu(\mu(i_*\mathbb{C}_{\triangle^\circ_{-\vec{c}}})) \cong \mu(\mathcal{R}(i_*\mathbb{C}_{\triangle^\circ_{-\vec{c}}})) \cong \mu(i_*\mathbb{C}_{\triangle^\circ_{\vec{c}}}).$$

The quasi-isomorphism $\mu \circ \mathcal{D} \cong \alpha \circ \mu$ induces

$$\alpha(\nu(\mathbb{L}_{-\vec{c},h^{-1}})) \cong \alpha(\mu(i_*\mathbb{C}_{\triangle^\circ_{\vec{c}}})) \cong \mu(\mathcal{D}(i_*\mathbb{C}_{\triangle^\circ_{\vec{c}}})) \cong \mu(i_!\omega_{\triangle^\circ_{\vec{c}}}).$$

It is easy to see that $\alpha(\nu(\mathbb{L}_{-\vec{c},h^{-1}})) = \mathbb{L}_{\vec{c},h}$. Therefore we have

$$\mathbb{L}_{\vec{c},h} \cong \mu(i_!\omega_{\triangle^\circ_{\vec{c}}}).$$

□

Appendix A. Review of Geometric Categories and Proof of Proposition 5.6

A.1. **Review of analytic-geometric categories.** We recall definitions and basic properties from [vdDM].

Definition A.1 (analytic-geometric category)**.** We say that an *analytic-geometric category* $\mathcal{C}$ is given if each manifold X is equipped with a collection $\mathcal{C}(X)$ of subsets of X such that the following conditions are satisfied for all manifolds X and Y:

AG1. $\mathcal{C}(X)$ is a Boolean algebra of subsets of X, with $X \in \mathcal{C}(M)$.
AG2. If $A \in \mathcal{C}(X)$, then $A \times \mathbb{R} \subset \mathcal{C}(X \times \mathbb{R})$.
AG3. If $f : X \to Y$ is a proper analytic map and $A \in \mathcal{C}(X)$, then $f(A) \in \mathcal{C}(Y)$.
AG4. If $A \subset X$, and (U_i) is an open covering of X (i in some index set I), then $A \in \mathcal{C}(X)$ if and only if $A \cap U_i \in \mathcal{C}(U_i)$ for all $i \in I$.
AG5. Every bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

It is proved in [vdDM, Appendix D] that this indeed gives rise to a category $\mathcal{C}$. An object of $\mathcal{C}$ is a pair (A, X) with X a manifold and $A \in \mathcal{C}(X)$. A morphism $(A, X) \to (B, Y)$ is a continuous map $f : A \to B$ whose graph

$$\Gamma_f = \{(a, f(a)) \mid a \in A\} \subset A \times B$$

belongs to $\mathcal{C}(X \times Y)$. We usually refer to an object (A, X) of $\mathcal{C}$ as the $\mathcal{C}$-*set* A in X, or even just the $\mathcal{C}$-set A if its ambient manifold is clear from context. Similarly, a morphism $f : (A, X) \to (B, Y)$ is called a $\mathcal{C}$-*map* $f : A \to B$ if X and Y are clear from context.

The following basic properties are proved in [vdDM, Appendix D].

Theorem A.2. *Let X, Y be manifolds of dimension m, n, respectively, and let $A \in \mathcal{C}(X)$, $B \in \mathcal{C}(Y)$.*

(1) *Every analytic map $f : X \to Y$ is a $\mathcal{C}$-map.*
(2) *Given an open covering (U_i) of X, a map $f : A \to Y$ is a $\mathcal{C}$-map if and only if each restriction $f|_{U_i \cap A} : U_i \cap A \to Y$ is a $\mathcal{C}$-map.*
(3) *$A \times B \in \mathcal{C}(X \times Y)$, and the projections $A \times B \to A$ and $A \times B \to B$ are $\mathcal{C}$-maps.*
(4) *If $f : A \to Y$ is a proper $\mathcal{C}$-map and $Z \subset A$ is a $\mathcal{C}$-set, then $f(Z) \in \mathcal{C}(Y)$.*
(5) *If A is closed in X and $f : A \to Y$ is a $\mathcal{C}$-map, then $f^{-1}(B) \in \mathcal{C}(X)$.*
(6) *If $B_1, \ldots, B_k$ are $\mathcal{C}$-sets (in possibly different manifolds), then a map*

$$f = (f_1, \ldots, f_k) : A \to B_1 \times \cdots \times B_k$$

is a $\mathcal{C}$-map if and only if each $f_i : A \to B_i$ is a $\mathcal{C}$-map.
(7) *$cl(A), int(A) \in \mathcal{C}(X)$.*

Corollary A.3. *Assume $f : X \to Y$ is a $\mathcal{C}$-map which is also a homeomorphism. Then*

(1) *$f^{-1} : Y \to X$ is a $\mathcal{C}$-map.*
(2) *For any subset $A \subset X$, $A \in \mathcal{C}(X) \Leftrightarrow f(A) \in \mathcal{C}(Y)$.*

A.2. **Proof of Proposition 5.6.**

Proof. By (7) of Theorem A.2, it suffices to prove that

(1a) $\iota(\mathbb{L}_{-\vec{c},h^{-1}}) \in \mathcal{C}(D^*M_{\mathbb{R}})$,

(1b) $\mathbb{L}^\infty_{-\vec{c},h^{-1}} = \Lambda^\infty_{-\vec{c}}$.

Let

$$\begin{aligned} B(N_{\mathbb{R}}) &= \{y \in N_{\mathbb{R}} \mid |y|_{N_{\mathbb{R}}} < 1\}, \\ \bar{B}(N_{\mathbb{R}}) &= \{y \in N_{\mathbb{R}} \mid |y|_{N_{\mathbb{R}}} \leq 1\} = B(N_{\mathbb{R}}) \cup S(N_{\mathbb{R}}). \end{aligned}$$

Define

$$F : X_\Sigma \times \bar{B}(N_{\mathbb{R}}) \to M_{\mathbb{R}} \times \bar{B}(N_{\mathbb{R}}) = D^*M_{\mathbb{R}}, \quad (x, y) \mapsto (-\Phi_h(x), y).$$

Then F is a proper real analytic map. Let

$$L = \{(x, y) \in X_\Sigma \times B(N_{\mathbb{R}}) \mid x = j_0\Big(\frac{y}{\sqrt{1 - |y|^2_{N_{\mathbb{R}}}}}\Big)\}.$$

Let $\bar{L}$ be the closure of L in $\overline{B}(N_{\mathbb{R}})$, and let $L^\infty = \bar{L} \cap (X_\Sigma \times S(N_{\mathbb{R}}))$. By (4) of Theorem A.2, it suffices to prove that

(2a) $L \in \mathcal{C}(X_\Sigma \times \bar{B}(N_{\mathbb{R}}))$,

(2b) $L^\infty = \bigcup_{\tau \in \Sigma'} O^+_\tau \times ((-\tau^\circ) \cap S(N_{\mathbb{R}}))$.

Recall that $X_\sigma \cong \mathbb{C}^n$ for $\sigma \in \Sigma(n)$, and $\{X_\sigma \mid \sigma \in \Sigma(n)\}$ is an open cover of X_Σ. By AG4 of Definition A.1, it suffices to prove that, for any $\sigma \in \Sigma(n)$,

(3a) $L \cap (X_\sigma \times \bar{B}(N_{\mathbb{R}})) \in \mathcal{C}(X_\sigma \times \bar{B}(N_{\mathbb{R}}))$,

(3b) $L^\infty \cap (X_\sigma \times \bar{B}(N_{\mathbb{R}})) = \bigcup_{\tau \in \Sigma', \tau \subset \sigma} O^+_\tau \times ((-\tau^\circ) \cap S(N_{\mathbb{R}}))$.

Given $\sigma \in \Sigma(n)$, there exists a $\mathbb{Z}$-basis $\{w_1, \ldots, w_n\}$ of N such that

$$\{w_, \ldots, w_n\} \subset \{v_1, \ldots, v_r\},$$
$$\sigma = \{r_1 w_1 + \cdots + r_n w_n \mid r_j \geq 0\}.$$

Let $\{w_1^\vee, \ldots, w_n^\vee\}$ be the dual $\mathbb{Z}$-basis of M, so that

$$\sigma^\vee = \{s_1 w_1^\vee + \cdots + s_n w_n^\vee \mid s_j \geq 0\}.$$

We have

$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[\chi^{w_1^\vee}, \ldots, \chi^{w_n^\vee}].$$

Let $Z_j = \chi^{w_j^\vee}$. Then $Z_1, \ldots, Z_n$ are holomorphic coordinates of

$$X_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}^n.$$

The image of $j_0 : N_{\mathbb{R}} \to X_\Sigma$ is contained in X_σ, and j_0 is given by

$$y \mapsto (e^{\langle w_1^\vee, y\rangle}, \ldots, e^{\langle w_n^\vee, y\rangle}),$$

or equivalently,

$$\sum_{j=1}^n y_j w_j \mapsto (e^{y_1}, \ldots, e^{y_n}).$$

Let $(\ ,\ )_{N_{\mathbb{R}}}$ denote the inner product on $N_{\mathbb{R}}$, and let $g_{ij} = (w_i, w_j)_{N_{\mathbb{R}}}$. Define $Q : \mathbb{R}^n \to \mathbb{R}$ by

$$Q(y_1, \ldots, y_n) \stackrel{\text{def}}{=} \Big|\sum_{j=1}^n y_j w_j\Big|^2_{N_{\mathbb{R}}} = \sum_{j,k=1}^n g_{jk} y_j y_k.$$

Define

$$\bar{B} = \{y \in \mathbb{R}^n \mid Q(y) \leq 1\}, \quad S = \{y \in \mathbb{R}^n \mid Q(y) = 1\}.$$

Then $\bar{B}$ is a solid ellipsoid in $\mathbb{R}^n$. Define

$$\psi : \mathbb{R}^n \times \bar{B} \longrightarrow X_\sigma \times \bar{B}(N_{\mathbb{R}}), \quad (x, y) \mapsto (x, \sum_{j=1}^n y_j w_j)$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then ψ is an injective, proper, real analytic map. Define

$$\begin{aligned} L_1 := \psi^{-1}(L) &= \{(x, y) \in \mathbb{R}^n \times \bar{B} \mid Q(y) < 1,\ x_i = \exp\Big(\frac{y_i}{\sqrt{1 - Q(y)}}\Big)\} \\ &= \Big\{(x, y) \in \mathbb{R}^n \times \bar{B} \mid Q(y) < 1,\ x_j > 0,\ \frac{y_i}{\log x_i} = \sqrt{1 - Q(y)}\Big\}. \end{aligned}$$

Let $\bar{L}_1$ be the closure of L_1 in $\mathbb{R}^n \times \bar{B}$, and let $L_1^\infty = \bar{L}_1 \cap (\mathbb{R}^n \times S)$. Then

$$\psi(L_1) = L, \quad \psi(\bar{L}_1) = \bar{L}, \quad \psi(L_1^\infty) = L^\infty.$$

So it suffices to prove that

(4a) $L_1 \in \mathcal{C}(\mathbb{R}^n \times \bar{B})$,
(4b) $L_1^\infty = \{(x, y) \in \mathbb{R}^n \times S \mid y_j \leq 0, \quad x_j \geq 0, \quad x_1 y_1 = \cdots = x_n y_n = 0\}$.

Given any subset $I \subset \{1, 2, \ldots, n\}$, define

$$U_I = \{(x_1, \ldots, x_n) \mid |x_i| < 1 \text{ for } i \in I,\ |x_i| > \frac{1}{2} \text{ for } i \notin I\}.$$

Then $\{U_I \mid I \subset \{1, \ldots, n\}\}$ is an open cover of $\mathbb{R}^n$, and $\{U_I \times \bar{B} \mid I \subset \{1, \ldots, n\}\}$ is an open cover of $\mathbb{R}^n \times \bar{B}$. By AG4 of Definition A.1, it suffices to prove that, for any $I \subset \{1, \ldots, n\}$,

(5a) $L_1 \cap (U_I \times \bar{B}) \in \mathcal{C}(U_I \times \bar{B})$.
(5b)

$$\begin{aligned} L_1^\infty \cap (U_I \times \bar{B}) &= \{(x, y) \in U_I \times S \mid x_j \geq 0 \text{ for } j = 1, \ldots, n \\ &\qquad y_i \leq 0 \text{ and } x_i y_i = 0 \text{ for } i \in I, \quad y_i = 0 \text{ for } i \notin I\} \end{aligned}$$

Without of loss of generality, we assume that $I = \{1, 2, \ldots, d\}$, where $0 \leq d \leq n$. (In particular, I is empty when $d = 0$.) The other cases can be obtained by permutations of $\{1, \ldots, n\}$.

Let $J = (-\infty, -1/2) \cup (1/2, \infty)$. Then $U_I = (-1, 1)^d \times J^{n-d}$. Define

$$\begin{aligned} \phi : \mathbb{R}^d \times J^{n-d} \times \bar{B} &\longrightarrow U_I \times \bar{B} \\ ((t_1, \ldots, t_d), (x_{d+1}, \ldots, x_n), y) &\mapsto ((f(t_1), \ldots, f(t_d)), (x_{d+1}, \ldots, x_n), y). \end{aligned}$$

Then ϕ is a homeomorphism, and both ϕ and ϕ^{-1} are $\mathcal{C}$-maps. To prove (5a), it suffices to prove that

$$\phi^{-1}(L_1 \cap (U_I \times \bar{B})) \in \mathcal{C}(\mathbb{R}^d \times J^{n-d} \times \bar{B}).$$

We will prove that

$$\phi^{-1}(L_1 \cap (U_I \times \bar{B})) \in \mathcal{C}_{an}(\mathbb{R}^d \times J^{n-d} \times \bar{B}).$$

We have

$$\begin{aligned}
&\phi^{-1}(L_1 \cap (U_I \times \bar{B})) \\
= \ &\{((t_1, \dots, t_d), (x_{d+1}, \dots, x_n), y) \in \mathbb{R}^d \times J^{n-d} \times \bar{B} \mid t_i > 0, x_i > \frac{1}{2}, Q(y) < 1, \\
&\sqrt{1 - Q(y)} = -t_i y_i, \ 1 \le i \le d; \ \log x_i \sqrt{1 - Q(y)} = y_i, \ d+1 \le i \le n\} \\
= \ &\{((t_1, \dots, t_d), (x_{d+1}, \dots, x_n), y) \in \mathbb{R}^d \times J^{n-d} \times \bar{B} \mid t_i > 0, x_i > \frac{1}{2}, \\
&y_1, \dots, y_d < 0, \quad (\log x_{d+1}) y_{d+1}, \dots, (\log x_n) y_n \ge 0, \quad Q(y) < 1 \\
&1 - Q(y) = t_i^2 y_i^2, \ 1 \le i \le d; \quad (\log x_i)^2 (1 - Q(y)) = y_i^2, \ d+1 \le i \le n\}
\end{aligned}$$

$\phi^{-1}(L_1 \cap (U_I \times \bar{B}))$ is defined by equalities and inequalities of real analytic functions, so

$$\phi^{-1}(L_1 \cap (U_I \times \bar{B})) \in \mathcal{C}_{an}(\mathbb{R}^d \times J^{n-d} \times \bar{B}).$$

This proves (5a).

Note that

$$\begin{aligned}
&\phi^{-1}(\bar{L}_1 \cap (U_I \times \bar{B})) \\
= \ &\{((t_1, \dots, t_d), (x_{d+1}, \dots, x_n), y) \in \mathbb{R}^d \times J^{n-d} \times \bar{B} \mid t_i \ge 0, x_i > \frac{1}{2}, \\
&y_1, \dots, y_d \le 0, \quad (\log x_{d+1}) y_{d+1}, \dots, (\log x_n) y_n \ge 0, \\
&1 - Q(y) = t_i^2 y_i^2, \ 1 \le i \le d; \quad (\log x_i)^2 (1 - Q(y)) = y_i^2, \ d+1 \le i \le n\}
\end{aligned}$$

which is a $\mathcal{C}_{an}$-set in $\mathbb{R}^d \times J^{n,d} \times \bar{B}$.

$$\begin{aligned}
&\phi^{-1}(L_1^\infty \cap (U_I \times \bar{B})) \\
= \ &\{((t_1, \dots, t_d), (x_{d+1}, \dots, x_n), y) \in \mathbb{R}^d \times J^{n-d} \times S \mid t_i \ge 0, x_i > \frac{1}{2}, \\
&y_1, \dots, y_d \le 0, \ t_i y_i = 0, \ 1 \le i \le d; \quad y_i = 0, \ d+1 \le i \le n\}.
\end{aligned}$$

This proves (5b). ☐

Appendix B. Generating sets of line bundles

The Lagrangians that generate our Fukaya category are T-dual to equivariant ample line bundles. In this section we will show that such line bundles also generate the category of equivariant coherent sheaves. The theorem we are after is a slight generalization of a theorem of Seidel:

Theorem B.1 (Seidel). *If X is smooth and projective, then $\mathcal{P}\mathrm{erf}_T(X)$ is generated by line bundles.*

Proof. The proof of [Ab2, Proposition 1.3] shows that $\mathcal{P}\mathrm{erf}(X)$ is generated by line bundles. The same proof works in the T-equivariant setting, by the following observation. Given a T-equivariant coherent sheaf $\mathcal{F}$, there exists a T-equivariant ample line bundle $\mathcal{L}$ such that the underlying nonequivariant coherent sheaf $\mathcal{F} \otimes \mathcal{L}$ is generated by global sections. The T-action on $\mathcal{F} \otimes \mathcal{L}$ induces a T-action on $H^0(X, \mathcal{F} \otimes \mathcal{L})$. There exists a basis $s_1, \dots, s_N$ of $H^0(X, \mathcal{F} \otimes \mathcal{L})$ and characters $\chi_1, \dots, \chi_N \in \mathrm{Hom}(T, \mathbb{C}^*)$ such that $t \cdot s_i = \chi_i(t) s_i$ for all $t \in T$. Then $s_1, \dots, s_N$

defines a surjective morphism $\oplus_{i=1}^{N}\mathcal{L}^{-1}\otimes\mathcal{O}_X(\chi_i)\to\mathcal{F}$ of T-equivariant coherent sheaves, where $\mathcal{O}_X(\chi_i)$ is the structure sheaf equipped with the T-equivariant structure given by the character χ_i. □

The stronger version we prove is the following:

Theorem B.2. *If X is smooth and projective, then $\mathcal{P}\mathrm{erf}_T(X)$ is generated by T-equivariant ample line bundles.*

Proof. Let $\mathcal{A}$ be the full triangulated dg sub-category of $\mathcal{P}\mathrm{erf}_T(X)$ generated by T-equivariant ample line bundles. We need to show that $\mathcal{A}=\mathcal{P}\mathrm{erf}_T(X)$. We may see that $\mathcal{A}$ is a full, dense triangulated subcategory of $\mathcal{P}\mathrm{erf}_T(X)$ by the same argument used in the proof of Theorem B.1 given in [Ab2]. (Recall that a triangulated subcategory is called *dense* if every object is a direct summand of an object in the subcategory.)

Now by [Th, Theorem 2.1], to show that $\mathcal{A}=\mathcal{P}\mathrm{erf}_T(X)$ it suffices to show that the subgroup $K(\mathcal{A})$ of $K(\mathcal{P}\mathrm{erf}_T(X))=K_T(X)$ is equal to $K_T(X)$. We will show that $K_T(X)$ is additively generated by T-equivariant ample line bundles.

Let $r=|\Sigma(1)|$ be the number of 1-dimensional cones, and let $v=|\Sigma(n)|$ be the number of maximal cones, which is also equal to the number of T-fixed points in X. Then $r=\mathrm{rank}_{\mathbb{Z}}\mathrm{Pic}_T(X)$.

Step 1. Claim: There exists a $\mathbb{Z}$-basis $\{L_1,\dots,L_r\}$ of $(\mathrm{Pic}_T(X),\otimes)$ such that $L_1,\dots,L_r$ are T-equivariant ample line bundles.

There exists a primitive ample class $\alpha\in H^{1,1}(X;\mathbb{Z})$. Let M_1 be a T-equivariant line bundle with $c_1(M_1)=\alpha$. There exist T-equivariant line bundles $M_2,\dots,M_r$ such that $\{M_1,\dots,M_r\}$ is a $\mathbb{Z}$-basis of $(\mathrm{Pic}_T(X),\otimes)$. There exist positive integers $n_2,\dots,n_r$ such that $M_i\otimes M_1^{\otimes n_i}$ are ample, $i=2,\dots,r$. Let

$$L_1=M_1,\quad L_i=M_i\otimes M_1^{\otimes n_i}\text{ for } i=2,\dots,r.$$

Then $\{L_1,\dots,L_r\}$ is the desired $\mathbb{Z}$-basis of $(\mathrm{Pic}_T(X),\otimes)$.

Step 2. Let $e_i=(c_1)_T(L_i)\in H^2_T(X;\mathbb{Z})$. Let $x_1,\dots,x_v$ be the T-fixed points of X, and let $\epsilon_j:x_j\to X$ be the inclusion. Let

$$u_{ij}=\epsilon_j^*e_i\in H^2_T(x_j;\mathbb{Z})\cong M.$$

Let $r_j:X\to x_j$ be the constant map. This gives rise to $r_j^*:H^2_T(x_j;\mathbb{Z})\cong M\to H^2_T(X;\mathbb{Z})$. Therefore we may view $u_{ij}\in M$ as elements in $H^2_T(X;\mathbb{Z})$. The map $\epsilon_j^*\circ r_j^*:M\to M$ is the identity map. By localization, for $i=1,\dots,r$, we have the following relation in $H^*_T(X)$:

$$\prod_{j=1}^{v}(e_i-u_{ij})=0.$$

Let V_{ij} be the T-equivariant line bundle with $(c_1)_T(V_{ij})=-u_{ij}$. It is a T-equivariant lifting of the trivial holomorphic line bundle $\mathcal{O}_X$. Define

$$y_{ij}=\mathrm{ch}_T(L_i\otimes V_{ij})=e^{e_i-u_{ij}},\quad i=1,\dots,r,\quad j=1,\dots,v.$$

Then we have

$$\prod_{j=1}^{v}(y_{ij}-1)=0\in H^*_T(X;\mathbb{Q}),\quad i=1,\dots,r,$$

so

$$\prod_{j=1}^{v}(L_i \otimes V_{ij} - 1) = 0 \in K_T(X), \quad i = 1, \ldots, r. \tag{33}$$

Step 3. By [Mo, Proposition 3], any element in $K_T(X)$ can be written as

$$\sum a_{m_1,\ldots,m_r} L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}, \tag{34}$$

where

(i) $m_1, \ldots, m_r, a_{m_1,\ldots,m_r}$ are integers, and
(ii) all but finitely many $a_{m_1,\ldots,m_r}$ are zero.

We may use (33) to rewrite (34) as

$$\sum b_{m_1,\ldots,m_r} L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}, \tag{35}$$

where

(i)' $m_1, \ldots, m_r \in \{1, 2, \ldots, v\}$ (in particular, the sum is finite), and
(ii)' $b_{m_1,\ldots,m_r} \in \mathbb{Z}[M]$, the representation ring of T.

Note that (i)' implies that, for any equivariant lifting V of the trivial holomorphic line bundle $\mathcal{O}_X$, $V \otimes L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}$ is ample. Therefore $K_T(X)$ is additively generated by T-equivariant ample line bundles. □

Appendix C. Relation to Other Work

C.1. **Seidel, Auroux-Katzarkov-Orlov.** The homological mirror symmetry proofs of Seidel [S1] and Auroux-Katzarkov-Orlov [AKO1, AKO2], formulated in the Fukaya-Seidel version of the mirror, make use of the fact that the mirror categories are generated by a finite collection of objects (Lagrangian thimbles). Studying the images of a generating set (such as $\mathcal{O}(-1)$, $\mathcal{O}$, $\mathcal{O}(1)$ for $\mathbb{P}^2$)) in different formulations of homological mirror symmetry leads to the conjecture that the thimbles are equivalent as objects to the T-dual branes associated to these line bundles. More generally, one should search for a proof that the dictionary between superpotential W_Σ (see Section C.2) and microlocal condition Λ_Σ leads to an equivalence of categories.

Example: The projective plane $\mathbb{P}^2$. The mirror Landau-Ginzburg model of $\mathbb{P}^2$ is $(\mathbb{C}^*)^2$ together with the superpotential $W = z_1 + z_2 + 1/z_1 z_2$. The three critical points are $(1,1), (w,w), (\bar{w},\bar{w})$ where $w = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, over the critical values $3, 3w, 3\bar{w}$ respectively. The Fukaya-Seidel category is a category of Lagrangian thimbles T_i together with directed perturbation when computing morphisms. These infinite Lagrangian branes T_i are the clockwise labeled vanishing thimbles over the positive-pointing rays λ_i starting from the critical values of W, parallel to the real axis. The difference between T-dual costandard/standard Lagrangians branes in $Fuk(T^*T_{\mathbb{R}}^\vee; \bar{\Lambda}_\Sigma)$ and Lagrangian thimbles in $FS((\mathbb{C}^*)^n, W)$ is illustrated in the following figure by looking at their images under the superpotential W. Indeed, the T-dual branes are very much like the vanishing thimbles in the case of $\mathbb{P}^2$: their images under W also propagate from the critical values, but in a "thickened" way.

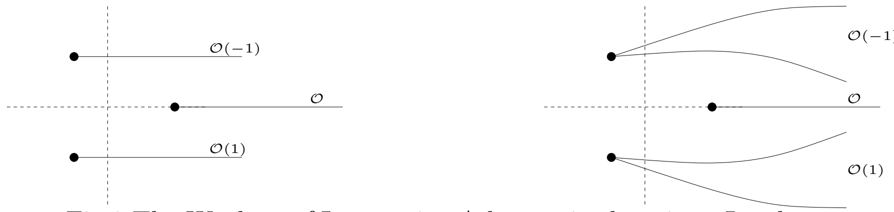

Fig.2 The W-plane of Lagrangian A-branes in the mirror Landau-Ginzburg model of $\mathbb{P}^2$. The images of the Lagrangian thimbles under the superpotential W are horizontal rays toward positive infinity, shown on the left. The images of the T-dual Lagrangians (with respect to $\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1)$) are shown on the right, which are the areas inside the curves. They are "thickened" versions of Lagrangian thimbles. Dashed lines are coordinate axis.

C.2. **Abouzaid.** Abouzaid studies the Fukaya category of the Landau-Ginzburg model dual to the toric variety. We will describe the construction in [Ab2] in our notation. We use the notation of Section 2.4. X_Σ is an n-dimensional smooth projective toric variety defined by a smooth complete fan $\Sigma \subset N_\mathbb{R}$, $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ is the set of 1-dimensional cones in Σ, and $v_i \in N$ is the generator of ρ_i, $i = 1, \ldots, r$.

Let $P \subset N_\mathbb{R}$ be the convex hull of $\{v_1, \ldots, v_r\}$. The Landau-Ginzburg model dual to X_Σ is a pair $((\mathbb{C}^*)^n, W)$, where $W : (\mathbb{C}^*)^n \to \mathbb{C}$ is known as the superpotential. In our notation, W is a holomorphic function on $T^\vee = M \otimes \mathbb{C}^*$, the complex dual to the torus $T = N \otimes \mathbb{C}^*$ acting on X_Σ. Let $z^\alpha \in \mathrm{Hom}(T^\vee, \mathbb{C}^*)$ be the image of $\alpha \in N$ under the isomorphism $N \xrightarrow{\sim} \mathrm{Hom}(T^\vee, \mathbb{C}^*)$. The superpotential $W : T^\vee \to \mathbb{C}$ is a Laurent polynomial

$$W = \sum_{\alpha \in N} c_\alpha z^\alpha, \quad c_\alpha \in \mathbb{C}$$

with the constraint

$$\mathrm{Newton}(W) := \{\alpha \mid c_\alpha \neq 0\} = P. \tag{36}$$

Up to now, W depended only on the fan Σ. To apply tropical geometry, Abouzaid picks an ample line bundle $\mathcal{L}_\nu$ on X_Σ associated to a strictly convex piecewise linear function $\nu : N_\mathbb{R} \to \mathbb{R}$ and defines a 1-parameter family of superpotentials (recall that ν takes integral values on the lattice N):

$$W_t = \sum_{\alpha \in N} c_\alpha t^{-\nu(\alpha)} z^\alpha, \quad t \in \mathbb{C}^*,$$

where $\{c_\alpha\}$ are fixed constants satisfying (36). Therefore $((\mathbb{C}^*)^n, W_t)$ can be viewed as the dual of the polarized toric variety $(X_\Sigma, \mathcal{L}_\nu)$. $M_t = W_t^{-1}(0)$ is a smooth hypersurface in $(\mathbb{C}^*)^n$.

We have

$$T^\vee \cong M \otimes \mathbb{C}^* \cong (M_\mathbb{R}/M) \times M_\mathbb{R} \cong T(T^\vee_\mathbb{R})$$

Under the isomorphism $T^\vee \cong \mathbb{C}^*$, the projection $T^\vee_\mathbb{R} \times M_\mathbb{R} \to M_\mathbb{R}$ gets identified with the logarithm map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ in tropical geometry:

$$\mathrm{Log}(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|).$$

Let $\mathcal{A}_t := \mathrm{Log}(M_t)$ be the amoeba of M_t. When X_Σ is Fano, there is a unique *bounded* connected component Q_t of $\mathbb{R}^n - \mathcal{A}_t$. Abouzaid defines a pre-category of *tropical Lagrangian sections* whose objects (Lagrangian branes) are sections of

the restrictions of the logarithm moment map to Q_t; these Lagrangian branes are compact n-dimensional submanifolds of $T^\vee_{\mathbb{R}} \times M_{\mathbb{R}}$ with boundary in M_t.

We will describe the relation between the tropical version of Abouzaid's Lagrangian branes (see [Ab1, Section 3.3]) and ours. Let

$$\Pi = \lim_{t\to\infty} \frac{\mathcal{A}_t}{\log t} \subset M_{\mathbb{R}} \cong \mathbb{R}^n$$

be the tropical amoeba, let $Q \subset M_{\mathbb{R}}$ and $M_\infty \subset T(M_{\mathbb{R}}/M) \cong (\mathbb{C}^*)^n$ be the corresponding limits of Q_t and M_t as $t \to \infty$. Then Q is a connected component of $\mathbb{R}^n - \Pi$. Indeed, Q is the moment polytope of the ample line bundle $\mathcal{L}_\nu$. Let $\Phi_\nu : X_\Sigma \to M_{\mathbb{R}}$ be a moment map of $\mathcal{L}_\nu$, and let $\Psi_\nu = \Phi_\nu \circ j_0 : N_{\mathbb{R}} \to M_{\mathbb{R}}$. Define $\phi_\nu : M_{\mathbb{R}} \times N_{\mathbb{R}} \to T^\vee_{\mathbb{R}} \times M_{\mathbb{R}}$ by $\phi_\nu(x,y) = (p(x), \Psi_\nu(y))$, where $p : M_{\mathbb{R}} \to M_{\mathbb{R}}/M = T^\vee_{\mathbb{R}}$ is the natural projection. Given any line bundle $\mathcal{L}_{\vec{c}}$ over X_Σ,

$$L_{\nu,\vec{c},h} := \phi_\nu(\mathbb{L}_{\vec{c},h}) = \{(p \circ \Psi_{\vec{c},h}(y), \Psi_\nu(y)) \mid y \in N_{\mathbb{R}}\} \subset T^\vee_{\mathbb{R}} \times M_{\mathbb{R}}$$

is (in the equivalence class of) Abouzaid's Lagrangian brane associated to the line bundle $\mathcal{L}_{\vec{c}}$.

Abouzaid defined a relative Fukaya category $Fuk((\mathbb{C}^*)^n, M)$, where M is a fiber of $W : (\mathbb{C}^*) \to \mathbb{C}$, and proved that $\mathcal{L}_{\vec{c}} \mapsto L_{\nu,\vec{c},h}$ defines a full embedding

$$DCoh(X_\Sigma) \to D^\pi Fuk((\mathbb{C}^*)^n, M),$$

which is expected to be an equivalence when X_Σ is Fano. So when X_Σ is a smooth projective Fano toric variety, it is natural to expect

$$DCoh(X_\Sigma) \cong DFuk(T^*T^\vee_{\mathbb{R}}, \bar{\Lambda}_\Sigma) \cong D^\pi Fuk((\mathbb{C}^*)^n, M), \tag{37}$$

where the equivalences are given by $\mathcal{L}_{\vec{c}} \mapsto \bar{\mathbb{L}}_{\vec{c},h} \mapsto L_{\nu,\vec{c},h}$.

Abouzaid's work (as the authors understand it) is inspired in part by T-duality, but in it there is not the emphasis (as there is here) that T-duality is the precise mechanism for mirror symmetry, nor is there a connection to constructible sheaves.

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